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VARIABLE STRUCTURE OBSERVER DESIGN IN MATRIX SECOND-ORDER FORMULATION FOR LINEAR & NONLINEAR VIBRATING SYSTEMS USING VELOCITY MEASUREMENTS

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ABSTRACT

This paper addresses the problem of Variable Structure Systems (VSS) observer design based on the natural Matrix Second Order (MSO) model that represents a large class of linear and nonlinear mechanical vibrating structures. In this natural MSO form, the symmetric and definiteness properties of the system matrices are exploited to search for a suitable Lyapunov function and an effective VSS estimation law. The proposed observers can be used to robustly estimate oscillations in other degrees-of-freedom (*dof*) of a multiple-degrees-of-freedom (*mdof*) linear vibrating system by processing measured velocity signals and control inputs from one or more *dof* in the presence of matched uncertainties. The method is then extended to cover a class of *mdof* nonlinear vibrating structures with Lipschitz non-linearities. The benefits of this approach are that it does not require an initial modal transformation, and the observer design problem is solved without resorting to the solution of a nonlinear matrix Riccati equation.

INTRODUCTION

For a given dynamic system, the problem of observer design is to estimate the current states of the system based solely on available measured information [1]. Observer design in state-space formulation, where the dynamics of the physical plant are described by a set of first-order differential equations, provides a unified approach for solving a large class of estimation problems since most physical systems in practice can be modelled in state-space form [2-3]. In many applications, the designer simply transforms the relevant plant model into state-space form, and then uses a wealth of existing analysis and synthesis tools for observer design.

One of the key issues in designing state estimators is plant uncertainty [4]. In dealing with plant parameter variations and disturbance rejection, Variable Structure Systems (VSS) design methodology is a well-known solution due to its appealing property that renders the estimators invariant to matched disturbances [5]. VSS observer design theory for linear uncertain systems modelled in state-space has been developed in great depth. In general, there are two main approaches for designing VSS state estimators for linear systems. One approach, proposed by Utkin [6] is to transform the first-order state-space model into a control canonical form such that the system's output is combined with the state-variable vector. The *equivalent control* principle is then used to enforce a sliding mode in the estimation error space. In the other approach, initiated by Walcott and Zak [7], asymptotic convergence of the estimation error is studied via Lyapunov stability theory. Recently, another VSS design method, which combines Utkin's and Walcott-Zak's method, was proposed in [8]. When uncertainty is present, Utkin's observer gives bounded estimation error, whereas Walcott-Zak's observer guarantees asymptotic convergence of the error [9]. Although the design idea of Walcott-Zak's observer is appealing, it is of limited use in practice since it imposes strong structural constraint on the system matrices [10]. The combined approach proposed in [8] works under less conservative structural conditions. However, as stated by its authors, one difficulty is to identify when the states have moved to the sliding surface in order to correctly introduce the equivalent control signal. Introducing the control signal at the wrong time may result in incorrect estimation.

Observer design for nonlinear uncertain systems is still a difficult problem despite the high level of attention given to it by many researchers. A summary of current available approaches is presented in a review paper by Misawa and Hedric [11], while more recent techniques are reported in [12]. One common approach is to extend the classical Luenberger observer and Kalman filter technique for use in nonlinear systems. Safonov and Athans [13] considered a class of nonlinear systems for which an observer dynamic is linearisable, and proposed an extended Kalman filter of constant gain. They showed that such estimators have a high degree of robustness in terms of gain margin and phase margin when dealing with some specific nonlinearities. One way that allows one to find an observer with linearisable error dynamics is to apply a canonical transformation in state-space [14]. In this canonical transformation framework, high gain observers are designed to improve the convergence rate of the estimation error. However, unlike the linear transformation in linear models, it is difficult to find a suitable transformation for a general nonlinear system. Another more traditional Lyapunov method is to decompose the state-space model into a linear input-free part and a nonlinear state-dependent controlled part [15-17]. Observation of the linear part is then accounted for by using the conventional linear Luenberger observer technique, whereas the nonlinear part is treated via its Lipschitz constant. Developed in parallel but with a different objective for dealing with plant uncertainties, is the VSS observer design approach. In comparison with the above approaches, a VSS estimator has both a linear constant gain term, and a nonlinear switching term which functions much like a high gain and therefore has more potential for dealing with disturbances [18]. In this path, there are in general two main approaches for dealing with the system's nonlinearities and uncertainties. One approach is to combine the plant's nonlinear term with the disturbance input and to then treat this combination as a disturbance. This reduces the nonlinear observation problem to a linear one. The other approach is to treat the nonlinear term using its Lipschitz constant. Uncertainties and nonlinearities are then handled by a switching surface, the design of which usually requires a solution of a nonlinear Riccati equation [16], [19]. VSS observer design with measurement noise has also bean discussed by many authors, e.g., [18-20].

However, in dealing with *mdof* vibrating systems governed by Newton's second law, the modelling equations are obtained naturally in the matrix second order (MSO) form [21-22]. Converting these equations of motion into a first-order state-space model does not preserve the advantageous characteristics of the system matrices in the original equations, e.g., symmetry, definiteness, and sparsity. These current VSS state-space design methods do not exploit the special structures of the system matrices in the natural MSO form. Furthermore, the system's order in state-space form is doubled and the physical importance of each entry in the state vector may not be preserved.

In recognising the above drawbacks of the state-space design framework, one must consider increasing recent concerns about solving the control and estimation problems directly in the second-order form. Control system design in second-order formulation was discussed in [23-26]. The Kalman filtering and square-root estimation problems were solved in second-order form in [27-28], while the issues of controllability and observability for second-order models were addressed in [29-31]. However, in these MSO observer designs, robustness against plant uncertainties has not been discussed.

In this paper, a new robust observer design method for a class of linear and non-linear vibrating structures is presented. The robust estimation problem is solved through the use of VSS observers designed directly in the MSO form that takes advantage of the symmetric and definiteness properties of the system matrices.

The proposed VSS observers are capable of estimating the motion in one or more *dof* of an *mdof* system *where no measurement is available* using velocity signals from other *dof*. The class of mechanical structures under consideration is the so-called "smart structures" where sensing mechanism is achieved by means of piezoelectric sensors. The reason why velocity information is considered is justified as follows. Firstly, in smart structures velocity signals are available via integration of accelerometer signals [32]. Secondly, compared to acceleration and displacement feedbacks for active control purposes, velocity feedback is shown to be the most stable feedback type with respect to the time-delay introduced by the zero-order hold logic present in digital-to-analog converters [33]. Velocity feedback is also shown to be robust against the spill-over problem encountered in controlling flexible structures [34].

The paper is organised as follows. In Part I, the problem of VSS observer design for linear and Lipschitz non-linear vibrating systems in second-order formulation using velocity measurement is formulated and solved in an ideal noiseless measurement setting. The problem of measurements contaminated by noise is then addressed in Part II for the nonlinear case. The paper ends with two simulation examples for the evaluation of the proposed VS observers.

I. VSS OBSERVER DESIGN IN MSO FORMULATION WITH NOISELESS MEASUREMENTS

I.1 Variable Structure Observer Design For Linear Vibrating Systems.

The problem of VSS estimator design for linear second-order models with velocity output is formulated as follows. Consider a second-order linear vibrating system described by:

$$\begin{cases} \mathbf{M\ddot{x}} + \mathbf{C\dot{x}} + \mathbf{Kx} = \mathbf{Bu} + \mathbf{Df}, \\ \mathbf{Y} = \mathbf{Q\dot{x}}, \end{cases} \tag{1}$$

where $\mathbf{M}$, $\mathbf{C}$, $\mathbf{K} \in \Re^{nxn}$ are symmetric, constant matrices; and $\mathbf{x}$, $\mathbf{u} \in \Re^{n}$ are time-dependent state vector and control force respectively. In addition, $\mathbf{M}$ is positive definite, $\mathbf{C}$, $\mathbf{K}$ are at least semi-positive definite, $\mathbf{B} \in \Re^{nxm}$ is the control distribution matrix, and $\mathbf{Q} \in \Re^{nxn}$ is the output distribution matrix. The vector $\mathbf{f}(\mathbf{x}, \mathbf{u}, t)$: $\Re^{nx1} \times \Re^{nx1} \times \Re^{+} \rightarrow \Re^{n}$ represents non-linearities and unknown time-varying disturbances, while the disturbance distribution matrix $\mathbf{D}$ is assumed to be known. This second-order matrix equation can be used to model a large class of mechanical, electrical, thermal, and fluid structure interaction systems. For *mdof* vibrating systems, this model can describe lumped-parameter systems or approximations of distributed structures. Based solely on the output information, an observer can hence be designed to robustly estimate the state variables in the presence of uncertainty. We make the following assumptions.

Assumption 1: The output measurement matrix $\mathbf{Q}$ is symmetric.
Assumption 2: There exists a matrix $\mathbf{R}$ such that:

$$\mathbf{D} = \mathbf{QR}. \tag{2}$$

Assumption 3: There exists a matrix $\mathbf{H}_v$ such that: $\overline{\mathbf{C}} = \mathbf{C} + \mathbf{H}_v\mathbf{Q}$ is positive definite.

Assumption 4: Let: $\mathbf{Rf} = [\xi_1 \quad \xi_2 \quad \dots \quad \xi_n]$. Assume that ξ_i is bounded with known bound, i.e., there exists a scalar constant p such that:

$$p > \max_{1 \le i \le n} |\xi_i|. \tag{3}$$

Remark 1. *Assumption 1* can be easily satisfied if the velocity of each *dof* in system (1) can be measured or derived independently. This implies that rank(Q)=q, where q is the total number of the generalised co-ordinates that appeared in the measured output. *Assumption 2* holds when the matching condition is satisfied, i.e., one sensor is needed for each unknown disturbance input. *Assumption 3* implies that the linear part of system (1) is stabilizable by linear velocity feedback. Therefore, these assumptions cover a fairly general class of mechanical vibrating systems. A VSS observer is proposed as follows.

Theorem 1. For the system (1) that satisfies *Assumptions 1, 2, 3* and *4*, the following observer guarantees asymptotic convergence of the first derivative of the estimation error to the origin:

$$\mathbf{M\ddot{\tilde{x}}} + \mathbf{C\dot{\tilde{x}}} + \mathbf{K\tilde{x}} + S_e - \mathbf{H}_v(\mathbf{Y}_v - \mathbf{Q\dot{\tilde{x}}}) = \mathbf{Bu}, \tag{4}$$

where: $\tilde{\mathbf{x}}$ is the estimate of $\mathbf{x}$,

$$S_e = -p\mathbf{Q}sign(\mathbf{Q\dot{e}}), \quad \mathbf{e} = \mathbf{x} - \tilde{\mathbf{x}}. \tag{5}$$

The proof of Theorem 1 is given in the Appendix.

I.2 Extension to Vibrating Systems with Lipschitz Non-linearity.

In this section, the above result is extended for a class of non-linear vibrating systems with Lipschitz non-linearity. System (1) is modified to include a non-linear term as follows:

$$\begin{cases} \mathbf{M\ddot{x}} + \mathbf{C\dot{x}} + \mathbf{Kx} + \mathbf{\Phi}(\mathbf{x}, \mathbf{\dot{x}}, \mathbf{u}) = \mathbf{Bu} + \mathbf{Df}, \\ \mathbf{Y} = \mathbf{Q\dot{x}}. \end{cases} \tag{6}$$

We make the following assumptions regarding the system's non-linearity.

Assumption 5. $\mathbf{\Phi}(\mathbf{x}, \mathbf{\dot{x}}, \mathbf{u})$ is a Lipschitz non-linearity with Lipschitz constant γ satisfying:

$$\left\| \mathbf{\Phi}(\mathbf{x}, \mathbf{\dot{x}}, \mathbf{u}) - \mathbf{\Phi}(\tilde{\mathbf{x}}, \dot{\tilde{\mathbf{x}}}, \mathbf{u}) \right\|_2 \le \gamma \left\| \mathbf{\dot{x}} - \dot{\tilde{\mathbf{x}}} \right\|. \tag{7}$$

Assumption 6. There exists a matrix $\mathbf{H}_v$ such that $\mathbf{E}_v = \mathbf{C} + \mathbf{H}_v\mathbf{Q} - \gamma\mathbf{I}$ is positive definite.

Remark 2. The non-linear quantity $\mathbf{\Phi}(\mathbf{x}, \mathbf{\dot{x}}, \mathbf{u})$ is not lumped with the disturbance $\mathbf{f}$ since the matching condition may no longer hold. The square and cubic non-linearity associated with non-linear spring of many mechanical structures can be regarded as Lipschitz non-linear, provided that displacement range and operating frequency is known. Therefore, system (6) represents a fairly large class of non-linear vibrating structures. Note that *Assumption 6* is more stringent than *Assumption 3*, in the sense that for non-linear systems, the fact that the linear part of system (6) is stabilisable by linear velocity feedback may not be sufficient to guarantee asymptotic convergence of the estimation error.

A VSS observer is proposed as follows.

Theorem 2. For system (6) that satisfies *Assumptions 5 & 6*, the following observer guarantees asymptotic convergence of the first derivative of the estimation error to the origin:

$$\mathbf{M}\ddot{\hat{\mathbf{x}}} + \mathbf{C}\dot{\hat{\mathbf{x}}} + \mathbf{K}\hat{\mathbf{x}} + \mathbf{\Phi}(\hat{\mathbf{x}}, \dot{\hat{\mathbf{x}}}, \mathbf{u}) + S_e - \mathbf{H}_v(\mathbf{Y} - \mathbf{Q}\dot{\hat{\mathbf{x}}}) = \mathbf{Bu} \cdot \qquad (8)$$

where S_e is designed as in (5).

The proof of Theorem 2 is given in the Appendix.

II. VARIABLE STRUCTURE OBSERVER DESIGN IN MATRIX SECOND-ORDER FORMULATION WITH MEASUREMENT NOISE

Consider the case when the velocity input to the observer is contaminated by noise. The non-linear system with measurement noise is described by:

$$\begin{cases} \mathbf{M}\ddot{\mathbf{x}} + \mathbf{C}\dot{\mathbf{x}} + \mathbf{K}\mathbf{x} + \mathbf{\Phi}(\mathbf{x}, \dot{\mathbf{x}}, \mathbf{u}) = \mathbf{Bu} + \mathbf{Df}, \\ \mathbf{Y} = \mathbf{Q}(\dot{\mathbf{x}} + \boldsymbol{\eta}), \end{cases} \qquad (9)$$

where the vector quantity $\boldsymbol{\eta}$ is the channel noise with unknown statistics, but is bounded with known bound:

$$\|\boldsymbol{\eta}\| < \bar{\eta}, \ \bar{\eta} > 0. \qquad (10)$$

The following observer is proposed:

$$\mathbf{M}\ddot{\hat{\mathbf{x}}} + \mathbf{C}\dot{\hat{\mathbf{x}}} + \mathbf{K}\hat{\mathbf{x}} + \mathbf{\Phi}(\hat{\mathbf{x}}, \dot{\hat{\mathbf{x}}}, \mathbf{u}) - S_n - \mathbf{H}(\mathbf{Y} - \mathbf{Q}\dot{\hat{\mathbf{x}}}) = \mathbf{Bu}, \qquad (11)$$

where:

$$S_n = \begin{cases} \dfrac{\mathbf{PQ}p(\mathbf{Y} - \mathbf{Q}\dot{\hat{\mathbf{x}}})}{\|\mathbf{Y} - \mathbf{Q}\dot{\hat{\mathbf{x}}}\|}, & \text{if } \|\mathbf{Y} - \mathbf{Q}\dot{\hat{\mathbf{x}}}\| > \varepsilon, \\ \dfrac{\mathbf{PQ}p(\mathbf{Y} - \mathbf{Q}\dot{\hat{\mathbf{x}}})}{\varepsilon}, & \text{if } \|\mathbf{Y} - \mathbf{Q}\dot{\hat{\mathbf{x}}}\| \le \varepsilon, \end{cases} \qquad (12)$$

where the scalar constant p satisfies (3) and $\mathbf{P}$ is designed such that:

$\mathbf{E}_v + \dfrac{p\mathbf{PQQ}}{\boldsymbol{\varepsilon}}$ is positive definite.

Remark 4. A boundary layer switching scheme similar to (12) is employed in [16] and [20] to impose a bound on the estimation error for state-space VS observers with noisy measurement. Here the boundary layer technique is employed to ensure a non-divergence of the estimation error for a class of non-linear plants modelled in second-order formulation as follows.

Theorem 3. For system (9) satisfying *Assumptions 5 & 6*, observer (12) guarantees uniformly ultimate boundedness of the first derivative of the estimation error, i.e., there exists a scalar constant $\delta > 0$ and a Lyapunov function V such that $\dot{V} < 0$ whenever $\|\dot{\mathbf{e}}\| > \delta$.

The proof of Theorem 3 is given in the Appendix.

III. SIMULATION EXAMPLES.

In this section, performance of the proposed VSS observers is evaluated via two simulation examples.

Example 1. A 3-*dof* linear vibrating system.
Consider a three-layer instrumentation rack modelled as a 3-*dof* structure as shown in Figure 1. The disturbance force *f* is assumed

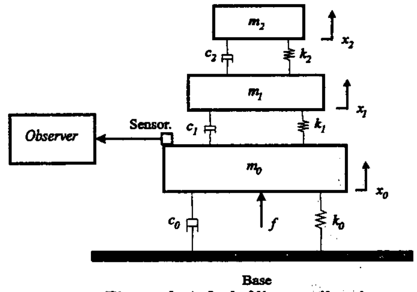

Figure 1. A 3-*dof* linear vibrating system.

to act on the base layer, i.e., primary mass m_0, fitted with an accelerometer. The vibration of the other two layers, i.e., masses m_1 and m_2 is estimated by the VSS observer (4). The following scenario is selected for simulation:
[m_0, c_0, k_0]=[5kg, 50/Kgm^2, 160kN/m],
[m_1, c_1, k_1]=[1kg, 10/Kgm^2, 106kN/m], and
[m_2, c_2, k_2]=[0.5kg, 10/Kgm^2, 96kN/m].

The equation of motion is then:

$$\begin{cases} \mathbf{M}\ddot{\mathbf{x}} + \mathbf{C}\dot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{B}u + \mathbf{D}f, \\ \mathbf{Y} = \mathbf{Q}\dot{\mathbf{x}}. \end{cases} \qquad (13)$$

where:

$$\mathbf{M} = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0.5 \end{bmatrix}, \ \mathbf{C} = \begin{bmatrix} 60 & -10 & 0 \\ -10 & 20 & -10 \\ 0 & -10 & 10 \end{bmatrix}, \ \mathbf{Q} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{K} = \begin{bmatrix} 2.66 & -1.06 & 0 \\ -1.06 & 2.02 & -0.96 \\ 0 & -0.96 & 0.96 \end{bmatrix} \times 10^5, \ \mathbf{B} = \mathbf{D} = [1 \ \ 0 \ \ 0]^T.$$

Assume that the 3-*dof* system is control-input free, i.e., $\mathbf{u} = \mathbf{0}$. The structure has 3 resonant modes at f_1=148rad/s, f_2=298rad/s, and f_3=580rad/s. Mass m_0 of the system is excited by a cyclic force of the form:

$$f(t) = \sin(148t) + \sin(298t) + \sin(580t) \qquad (13)$$

The velocity of mass m_0 is derived by single integration of the accelerometer output. The proposed VSS observer has the form:

$$\mathbf{M}\ddot{\hat{\mathbf{x}}} + \mathbf{C}\dot{\hat{\mathbf{x}}} + \mathbf{K}\hat{\mathbf{x}} + p\mathbf{Q}sign(\mathbf{Q}\dot{\mathbf{e}}) + \mathbf{H}_v(\mathbf{Y}_v - \mathbf{Q}\dot{\hat{\mathbf{x}}}) = \mathbf{Bu}, \qquad (14)$$

where $\mathbf{H}_v$ is an identity matrix of dimension 2×2. The observer model (14) is assumed to have the following modelling error: the mass matrix $\mathbf{M}$ and the stiffness matrix $\mathbf{K}$ are perturbed within ± 5% of their corresponding nominal values, while the damping matrix $\mathbf{C}$ (which is more difficult to model) is perturbed within ± 10% of its nominal value. In addition, a low frequency sinusoidal noise of 10 Hz (typical of the vibration present in a vehicle, aircraft, etc.) is added to the measured acceleration of the base layer. The added noise amplitude is equal to 3% of the measured acceleration.

Figures 2 and 3 show the respective estimated velocities and displacements of layers 1 and 2. The disturbance force acts on mass m_o, and the estimator is turned on at time t=0.2 sec. As shown in these figures, nearly perfect state replications are achieved within 0.05 sec. The simulations were run using a boundary layer technique [4], with switching gain p=10 and a boundary layer thickness of 0.001.

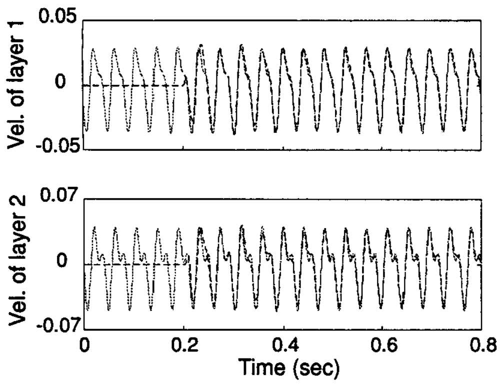

Figure 2. Velocity estimation for linear system with multi-sinusoidal excitation and unmatched uncertainty.

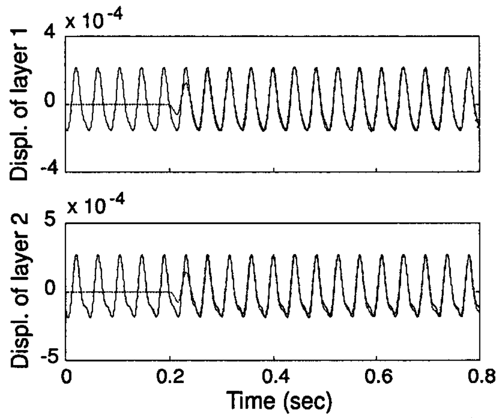

Figure 3. Displacement estimation for linear system with multi-sinusoidal excitation and unmatched uncertainty.

Example 2. A 2-*dof* nonlinear coupled oscillator.

Non-linear oscillations with cubic non-linearity can be found in many flexible mechanical structures where stretching and bending are significant, or in systems comprised of lumped-masses connected together via non-linear springs and/or dampers [35].

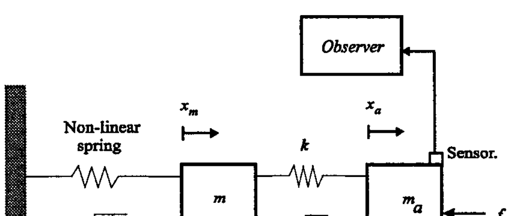

Figure 4. A 2-*dof* non-linear coupled oscillator with cubic non-linearity.

For a non-linear coupled oscillator with cubic non-linearity shown in Figure 4, the motion can be described in second-order formulation as follows:

$$\begin{cases} \mathbf{M}\ddot{\mathbf{x}} + \mathbf{C}\dot{\mathbf{x}} + \mathbf{K}\mathbf{x} + \phi(\mathbf{x}) = \mathbf{D}\mathbf{f}, \\ \mathbf{Y} = \mathbf{Q}\dot{\mathbf{x}}. \end{cases}$$

where:

$$\mathbf{M} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \mathbf{C} = \begin{bmatrix} 10 & -5 \\ -5 & 5 \end{bmatrix}, \mathbf{K} = \begin{bmatrix} 200 & -100 \\ -100 & 100 \end{bmatrix},$$

$$\mathbf{x} = [x \ \ x_a]^T,$$

$$\phi(\mathbf{x}) = \begin{bmatrix} 1000\left(\dfrac{x}{0.33}\right)^3 \\ 0 \end{bmatrix}, \ \mathbf{D} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \ \mathbf{Q} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

The mass matrix is in Kg, the damping matrix is per Kgm^2, and the stiffness matrix is in N/m. The coupled oscillators have two resonances at 6.2rad/s and 17rad/s. Assume the disturbance input is bounded within 20N (peak-peak), in the frequency range [3rad/s, 20rad/s], the Lipschitz constant is estimated to be γ=8. The velocity of mass m_a is derived by integrating the accelerometer signal. Velocity and displacement of mass m can be estimated by the following observer:

$$\mathbf{M}\ddot{\tilde{\mathbf{x}}} + \mathbf{C}\dot{\tilde{\mathbf{x}}} + \mathbf{K}\tilde{\mathbf{x}} + \phi(\tilde{\mathbf{x}}) - p\mathbf{Q}sign(\mathbf{Q}\dot{\mathbf{e}}) - \mathbf{H}_v\left(\mathbf{Y} - \mathbf{Q}\dot{\tilde{\mathbf{x}}}\right) = \mathbf{0},$$

where $\mathbf{H}_v = \mathbf{P} = \mathbf{I}_{2\times2}$. The observer model has a 5% error in the mass matrix **M** and the stiffness matrix **K**, and a 10% error in the damping matrix **C**. In addition, the accelerometer output from mass m_a is corrupted by random noise due to measurement imperfection. A white noise signal with a peak amplitude of 3% of the nominal acceleration is now added to the accelerator output. The system is excited by a cyclic force of the form:

$$f(t) = 2.5\sin(6.2t) + 2\sin(17t).$$

Figure 5 shows the estimated velocity and displacement of mass m. As seen from the graph, system state replication is achieved in 0.5 seconds. The observer is still capable of visually tracking the displacement and velocity of mass m in the presence of modelling uncertainty and disturbance. The simulation was run using a boundary layer technique with switching gain p=1 and a boundary layer thickness of 0.001.

Remark 2. For this nonlinear case study, the estimation problem cannot be solved using either the equivalent control approaches (Utkin's observer and the combined Utkin/Walcott-Zak's observer), or Walcott-Zak's approach. This is because the uncertainty and non-linearity terms in these observer design methods, are lumped together, such that the lumped uncertainties appear in two different measurement channels, i.e., one in the measurement of x and one in the measurement of x_a. Hence, two sensors are required, i.e., one for each uncertain channel. In addition, the proposed VSS observer does not rely either on the decision time for the injection of the equivalent control signal (which is difficult to determine in practice), or on the solution of two Lyapunov matrices (as for the Walcott-Zak approach) whose computation is impractical without symbolic manipulation software. Compared with other nonlinear design approaches that consider the Lipschitz constant in the state-space framework, the proposed observer does not encounter the nonlinear matrix Riccati equations, which is an advantage in dealing with vibrating systems that possess a high number of *dof*, for instance, the design of large space structures.

Remark 3. The proposed observers ensure asymptotic convergence of the estimated acceleration and velocity along those *dof* where *no measurements are available.* If the stiffness matrix **K** is positive-definite, then there are no rigid body modes and therefore no translational motions in the system [21]. In this case, estimation of the displacement can be simply obtained by single integration of the estimated velocity signals. If $\mathbf{K}$ is positive semi-definite, then displacement measurement is necessary for the estimation of all system state variables.

SUMMARY

A new VSS observer design method based on the natural second-order form of the modelling equation has been proposed in this paper. The objective of the new method is to exploit the definiteness and symmetric properties of the system matrices in their second-order formulations to search for a suitable Lyapunov function and an effective switching law. It is demonstrated that the non-linear matrix Riccati equation encountered in most VS observer designs via the Lyapunov method in the traditional first-order state-space form does not appear in the second-order design framework. Once the linear part of the dynamic system concerned is stabilised by linear velocity feedback, a VS observer can be readily synthesised.

APPENDIX

1. Proof of Theorem 1.

From (1) and (4), the dynamics of the estimation error are described by:

$$\mathbf{M}\ddot{\mathbf{e}}+\overline{\mathbf{C}}\dot{\mathbf{e}}+\mathbf{K}\mathbf{e}-S_e-\mathbf{Df}=\mathbf{0},\ \mathbf{e}=\mathbf{x}-\tilde{\mathbf{x}}.$$

Due the definiteness property of the system matrices $\mathbf{M}$ and $\mathbf{K}$, a simple Lyapunov function is formed as follows:

$$\mathbf{V}=\frac{1}{2}\left(\dot{\mathbf{e}}^T\mathbf{M}\dot{\mathbf{e}}+\mathbf{e}^T\mathbf{K}\mathbf{e}\right). \tag{A.1}$$

Taking the time derivative of this Lyapunov function gives:

$$\begin{aligned}\dot{\mathrm{V}} &=\frac{1}{2}\left[\left(\ddot{\mathbf{e}}^T\mathbf{M}\dot{\mathbf{e}}+\dot{\mathbf{e}}^T\mathbf{M}\ddot{\mathbf{e}}\right)+\left(\dot{\mathbf{e}}^T\mathbf{K}\mathbf{e}+\mathbf{e}^T\mathbf{K}\dot{\mathbf{e}}\right)\right]\\ &=\frac{1}{2}\left[(\mathbf{M}\ddot{\mathbf{e}})^T\dot{\mathbf{e}}+\dot{\mathbf{e}}^T\mathbf{M}\ddot{\mathbf{e}}\right]+\frac{1}{2}\left[(\mathbf{K}\mathbf{e})^T\dot{\mathbf{e}}+\mathbf{e}^T\mathbf{K}\dot{\mathbf{e}}\right]\\ &=\left[\mathbf{M}\ddot{\mathbf{e}}+\mathbf{K}\mathbf{e}\right]^T\dot{\mathbf{e}}\\ &=-\dot{\mathbf{e}}^T\overline{\mathbf{C}}\dot{\mathbf{e}}+\left[S_e+\mathbf{Df}\right]^T\dot{\mathbf{e}} &\text{(A.2)}\\ &=-\dot{\mathbf{e}}^T\overline{\mathbf{C}}\dot{\mathbf{e}}-\left[p\mathbf{Q}\,sign(\mathbf{Q}\dot{\mathbf{e}})-\mathbf{QRf}\right]^T\dot{\mathbf{e}}\\ &=-\dot{\mathbf{e}}^T\overline{\mathbf{C}}\dot{\mathbf{e}}-\left[p\,sign(\mathbf{Q}\dot{\mathbf{e}})-\mathbf{Rf}\right]^T(\mathbf{Q}\dot{\mathbf{e}})<0. &\text{(A.3)}\end{aligned}$$

Equality (A.2) is due to the symmetric property of $\mathbf{M}$ and $\mathbf{K}$, and inequality (A.3) follows directly from *Assumptions 3 & 4*. Therefore, $\dot{\mathrm{V}}<0$ whenever $\dot{\mathbf{e}}\neq\mathbf{0}$. This implies that the manifold $\dot{\mathbf{e}}=\mathbf{0}$ is reached asymptotically.

2. Proof of Theorem 2.

From (6) and (8), the dynamics of the estimation error are described by:

$$\mathbf{M}\ddot{\mathbf{e}}+\mathbf{C}\dot{\mathbf{e}}+\mathbf{K}\mathbf{e}+\left[\Phi(\mathbf{x},\dot{\mathbf{x}},\mathbf{u})-\Phi(\tilde{\mathbf{x}},\dot{\tilde{\mathbf{x}}},\mathbf{u})\right]-S_e+\mathbf{H}_v\mathbf{Q}\dot{\mathbf{e}}-\mathbf{Df}=\mathbf{0}.$$

Consider a Lyapunov function as in (A.1), its time derivative is then:

$$\begin{aligned}\dot{\mathbf{V}} &=\frac{1}{2}\left[\left(\ddot{\mathbf{e}}^T\mathbf{M}\dot{\mathbf{e}}+\dot{\mathbf{e}}^T\mathbf{M}\ddot{\mathbf{e}}\right)+\left(\dot{\mathbf{e}}^T\mathbf{K}\mathbf{e}+\mathbf{e}^T\mathbf{K}\dot{\mathbf{e}}\right)\right]\\ &=\left[\mathbf{M}\ddot{\mathbf{e}}+\mathbf{K}\mathbf{e}\right]^T\dot{\mathbf{e}}\\ &=-\left(\mathbf{C}\dot{\mathbf{e}}+\left[\Phi(\mathbf{x},\dot{\mathbf{x}},\mathbf{u})-\Phi(\tilde{\mathbf{x}},\dot{\tilde{\mathbf{x}}},\mathbf{u})\right]-S_e+\mathbf{H}_v\mathbf{Q}\dot{\mathbf{e}}-\mathbf{Df}\right)^T\dot{\mathbf{e}}\\ &=-\dot{\mathbf{e}}^T\left[\mathbf{C}+\mathbf{H}_v\mathbf{Q}\right]^T\dot{\mathbf{e}}-\left[\Phi(\mathbf{x},\dot{\mathbf{x}},\mathbf{u})-\Phi(\tilde{\mathbf{x}},\dot{\tilde{\mathbf{x}}},\mathbf{u})\right]^T\dot{\mathbf{e}}+\left[S_e+\mathbf{Df}\right]^T\dot{\mathbf{e}}\\ &=-\dot{\mathbf{e}}^T\left[\mathbf{C}+\mathbf{H}_v\mathbf{Q}\right]\dot{\mathbf{e}}-\left[\Phi(\mathbf{x},\dot{\mathbf{x}},\mathbf{u})-\Phi(\tilde{\mathbf{x}},\dot{\tilde{\mathbf{x}}},\mathbf{u})\right]^T\dot{\mathbf{e}}\\ &\quad-\left[p\,sign(\mathbf{Q}\dot{e})-\mathbf{Rf}\right]^T(\mathbf{Q}\dot{e}).\end{aligned}$$

We have:

$$\begin{aligned}&-\dot{\mathbf{e}}^T\left[\mathbf{C}+\mathbf{H}_v\mathbf{Q}\right]\dot{\mathbf{e}}-\left[\Phi(\mathbf{x},\dot{\mathbf{x}},\mathbf{u})-\Phi(\tilde{\mathbf{x}},\dot{\tilde{\mathbf{x}}},\mathbf{u})\right]^T\dot{\mathbf{e}}\\ &\le-\dot{\mathbf{e}}^T\left[\mathbf{C}+\mathbf{H}_v\mathbf{Q}\right]\dot{\mathbf{e}}+\left\|\Phi(\mathbf{x},\dot{\mathbf{x}},\mathbf{u})-\Phi(\tilde{\mathbf{x}},\dot{\tilde{\mathbf{x}}},\mathbf{u})\right\|\,\|\dot{\mathbf{e}}\|\\ &\le-\dot{\mathbf{e}}^T\left[\mathbf{C}+\mathbf{H}_v\mathbf{Q}\right]\dot{\mathbf{e}}+\gamma\|\dot{\mathbf{e}}\|\,\|\dot{\mathbf{e}}\|\\ &\le-\dot{\mathbf{e}}^T\left(\mathbf{C}+\mathbf{H}_v\mathbf{Q}-\gamma\mathbf{I}\right)\dot{\mathbf{e}}=-\dot{\mathbf{e}}^T\mathbf{E}_v\dot{\mathbf{e}}.\end{aligned}$$

Then:

$$\dot{\mathrm{V}}\le-\dot{\mathbf{e}}^T\mathbf{E}_v\dot{\mathbf{e}}-\left[p\,sign(\mathbf{Q}\dot{\mathbf{e}})-\mathbf{Rf}\right]^T(\mathbf{Q}\dot{\mathbf{e}})<0. \tag{A.4}$$

Inequality (A.4) is due *Assumptions 3 & 6*. Therefore, $\dot{\mathrm{V}}<0$ whenever $\dot{\mathbf{e}}\neq\mathbf{0}$. This implies that the surface $\dot{\mathbf{e}}=\mathbf{0}$ is reached asymptotically.

3. Proof of Theorem 3.

From (9) and (11), the dynamics of the estimation error are described by:

$$\mathbf{M}\ddot{\mathbf{e}}+\mathbf{C}\dot{\mathbf{e}}+\mathbf{K}\mathbf{e}+\left[\Phi(\mathbf{x},\dot{\mathbf{x}},\mathbf{u})-\Phi(\tilde{\mathbf{x}},\dot{\tilde{\mathbf{x}}},\mathbf{u})\right]+S_n+\mathbf{H}_v\mathbf{Q}\dot{\mathbf{e}}+\mathbf{H}_v\mathbf{Q}\eta-\mathbf{Df}=\mathbf{0}.$$

Consider the same Lyapunov function as in (A.1), we have:

$$\begin{aligned}\dot{V} &=\frac{1}{2}\left[\left(\ddot{\mathbf{e}}^T\mathbf{M}\dot{\mathbf{e}}+\dot{\mathbf{e}}^T\mathbf{M}\ddot{\mathbf{e}}\right)+\left(\dot{\mathbf{e}}^T\mathbf{K}\mathbf{e}+\mathbf{e}^T\mathbf{K}\dot{\mathbf{e}}\right)\right]\\ &=\dot{\mathbf{e}}^T\left[-(\mathbf{C}+\mathbf{H}_v\mathbf{Q})\dot{\mathbf{e}}-\left(\Phi(\mathbf{x},\dot{\mathbf{x}},\mathbf{u})-\Phi(\tilde{\mathbf{x}},\dot{\tilde{\mathbf{x}}},\mathbf{u})\right)-S_n-\mathbf{H}_v\mathbf{Q}\eta+\mathbf{Df}\right]\\ &=-\dot{\mathbf{e}}^T(\mathbf{C}+\mathbf{H}_v\mathbf{Q})\dot{\mathbf{e}}-\dot{\mathbf{e}}^T\left(\Phi(\mathbf{x},\dot{\mathbf{x}},\mathbf{u})-\Phi(\tilde{\mathbf{x}},\dot{\tilde{\mathbf{x}}},\mathbf{u})\right)-\dot{\mathbf{e}}^T\left[S_n+\mathbf{H}_v\mathbf{Q}\eta-\mathbf{Df}\right]\\ &\le-\dot{\mathbf{e}}^T\mathbf{E}_v\dot{\mathbf{e}}-\dot{\mathbf{e}}^T S_n+\dot{\mathbf{e}}^T\left[-\mathbf{H}_v\mathbf{Q}\eta+\mathbf{Df}\right]\\ &\le-\dot{\mathbf{e}}^T\mathbf{E}_v\dot{\mathbf{e}}-\dot{\mathbf{e}}^T S_n+\|\dot{\mathbf{e}}\|d,\end{aligned}$$

where $d=\|\mathbf{H}_v\mathbf{Q}\|\overline{\eta}+\|\mathbf{Q}\|p$, and the norm of a matrix is defined as $\|\mathbf{A}\|=\sqrt{\max\left[\lambda\left(\mathbf{A}^T\mathbf{A}\right)\right]}$, with $\lambda\left(\mathbf{A}^T\mathbf{A}\right)$ denotes the set of eigenvalues of $\mathbf{A}^T\mathbf{A}$. There are two possibilities:

Case 1. When $\|\mathbf{Y}-\mathbf{Q}\dot{\tilde{\mathbf{x}}}\|\le\varepsilon$, we have:

$$\begin{aligned}\dot{\mathrm{V}} &\le-\dot{\mathbf{e}}^T\mathbf{E}_v\dot{\mathbf{e}}-\dot{\mathbf{e}}^T S_n+\|\dot{\mathbf{e}}\|d\\ &\le-\dot{\mathbf{e}}^T\mathbf{E}_v\dot{\mathbf{e}}-\dot{\mathbf{e}}^T\frac{\mathbf{PQ}p\left(\mathbf{Y}-\mathbf{Q}\dot{\tilde{\mathbf{x}}}\right)}{\varepsilon}+\|\dot{\mathbf{e}}\|d\\ &\le-\dot{\mathbf{e}}^T\mathbf{E}_v\dot{\mathbf{e}}-\dot{\mathbf{e}}^T\frac{p\mathbf{PQQ}\,\dot{\mathbf{e}}+p\mathbf{PQQ}\,\eta}{\varepsilon}+\|\dot{\mathbf{e}}\|d\\ &\le-\dot{\mathbf{e}}^T\left(\mathbf{E}_v+\frac{p\mathbf{PQQ}}{\varepsilon}\right)\dot{\mathbf{e}}+\left|-\dot{\mathbf{e}}^T\frac{p\mathbf{PQQ}\,\eta}{\varepsilon}\right|+\|\dot{\mathbf{e}}\|d\\ &\le-a\|\dot{\mathbf{e}}\|^2+b\|\dot{\mathbf{e}}\|,\end{aligned}$$

where:

$$a=\min\lambda\left(\mathbf{E}_v+\frac{p\mathbf{PQQ}}{\varepsilon}\right),\quad b=d+\frac{p\|\mathbf{PQQ}\|\overline{\eta}}{\varepsilon}.$$

Therefore, $\dot{V}<0$ whenever $\|\dot{\mathbf{e}}\|>\frac{b}{a}=\delta$.

Case 2. When $\|\mathbf{Y}-\mathbf{Q}\dot{\tilde{\mathbf{x}}}\|>\varepsilon$, we have:

$$\begin{aligned}\dot{V} &\le-\dot{\mathbf{e}}^T\mathbf{E}_v\dot{\mathbf{e}}-\dot{\mathbf{e}}^T S_n+\dot{\mathbf{e}}^T\left[-\mathbf{H}_v\mathbf{Q}\eta+\mathbf{Df}\right]\\ &\le-\min\lambda(\mathbf{E}_v)\|\dot{\mathbf{e}}\|^2+\|\dot{\mathbf{e}}^T S_n\|+\|\dot{\mathbf{e}}\|d\\ &\le-\min\lambda(\mathbf{E}_v)\|\dot{\mathbf{e}}\|^2+p\|\mathbf{PQ}\|\|\dot{\mathbf{e}}\|+\|\dot{\mathbf{e}}\|d\\ &\le-a\|\dot{\mathbf{e}}\|^2+b\|\dot{\mathbf{e}}\|,\end{aligned}$$

where: $a=\min\lambda(\mathbf{E}_v)$, $b=p\|\mathbf{PQ}\|+d$.

Therefore, $\dot{V}<0$ whenever: $\|\dot{\mathbf{e}}\|>\frac{b}{a}=\delta$

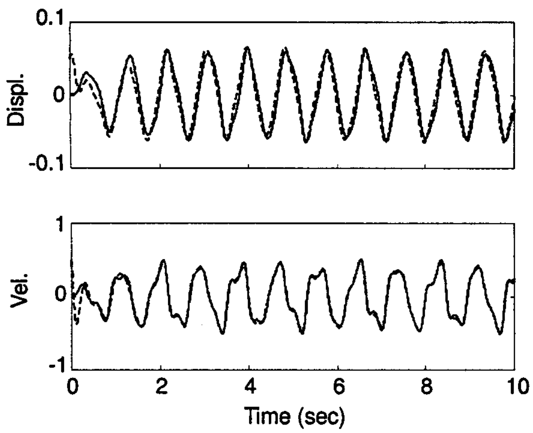

Figure 5. Estimation for nonlinear system with multi-sinusoidal excitation and unmatched uncertainty.

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