In this paper I will describe seven paradoxes, due to Zeno of Elea. I will show that they contain subtle arguments, not easily brushed aside. Resolution of the paradoxes in several cases requires nineteenth-century mathematics, which neither Zeno nor his contemporaries could have contemplated. In two cases, I will contend that the paradoxes are not solved even today.

1. Introduction

Zeno of Elea invented several paradoxes which are justly famous. They amount to interesting and plausible arguments for obviously false conclusions. This is the characteristic of a good paradox: it promises to be interesting and instructive. To say that Zeno’s arguments are plausible, is to say that it is not easy to describe where they go wrong in yielding absurd conclusions. This is a typical situation in philosophy! It leads us to re-examine what we imagined was absurd. Often we conclude that there was a core of truth in both positions, though uncovering those truths can be difficult.

In this paper, I am more interested in the paradoxes as being worthy of philosophical interrogation, than I am in historical detail. That is, I am more interested in what we can learn from them when we put the arguments in their strongest form, and less interested in what Zeno himself thought he was doing with them. That is another mark of a fecund idea: one which leads to later developments and distinctions. I aim to justify the great Bertrand Russell’s 1937 judgement of Zeno:

Having invented four arguments, all immeasurably subtle and profound, the grossness of subsequent philosophers pronounced him to be a mere ingenious juggler, and his arguments to be one and all sophisms (Russell, 1937:347).

Zeno (b. around 490 BCE) was a member of the Eleatic school. In this, he followed the view of Parmenides, who argued that all is one. In so doing he felt he had also to deny change. Zeno devised his paradoxes to support this position, and argued that our natural view, that there are such things as motion and spatial extension, is mistaken. That is, Zeno agreed with the conclusions of these paradoxes. But I said
earlier that these conclusions are absurd. That is the modern view. Nonetheless, as we will see, it takes at least nineteenth century mathematics to see the fallacies in Zeno’s arguments, so his arguments can be said to have survived 2,500 years and led to some very interesting insights. And even then, one (The Arrow) remains troublesome today.

Zeno’s paradoxes are variously numbered from four to forty. What explains this difference is that different classifications follow from different theoretical assessments of what is significant in each. I will classify them in my own way. I find seven, in three groups.

2. The Paradox of Plurality or Spatial Extension

If a thing had size, it would be many (because it would be spatially divisible). So, since there must be unity somewhere, things with size must be ultimately composed of things that have no size. But then, if existing things were composed of things that have no size, then the composite things can have no size either (because adding together things with no size cannot amount to having a non-zero size).

3. Four Paradoxes of Motion

3A: The Race Track I (also known as The Stadium). To move from place A to place B, one would have to move first half way between the two, then complete the journey. But to move to the half way point, one must first get to the quarter way point, and so on. Thus, to move from A to B one must complete an infinite number of distinct tasks, and an infinite number of distinct tasks cannot be completed in a finite time. Thus motion is contradictory and impossible.

3B: Achilles and the Tortoise. Achilles races the tortoise, and gives it a start. For Achilles to catch the tortoise, he runs first to where the tortoise starts from. The tortoise has moved on. Achilles continues to run to where the tortoise has moved to, but again the tortoise has moved on a little. Evidently, to catch the tortoise one must complete an infinite number of tasks, and as before one cannot do that in a finite time. Hence, Achilles cannot catch the tortoise.

3C: The Race Track II. Imagine three bodies broken down into their atomic parts, one body motionless, the others tracking past it in
opposite directions at the same speed. The time taken for each of the moving bodies
to go from one point to its adjacent point is fixed. The time taken for a given point
on one moving body to pass one on the other moving body is half that time, but that
is impossible because there is no such thing as half an instant.

3D: The Arrow. An arrow in flight is in the place that it is in. But that is indistingui-
shable from the arrow being motionless. (Freeze-frame the arrow at any in-
stant: it is motionless). How can a number of motionless things add up to motion?
Thus, the arrow is at rest.

4. Miscellaneous: Two Paradoxes

4A: Place. Suppose that existing things exist in space. That is, suppose that space
exists and it is what gives things their place: the place a thing is in, is defined as the
spatial point at which it exists. But then, since space exists, place exists. And since
whatever exists, exists at a place, it follows that place must be in a place also. Thus,
place is in a place is in a place, _ad infinitum._

4B: The Millet Seed. We tip a bushel of millet seed onto the ground, and we hear a
loud sound. Tip a single millet seed onto the ground, it makes no sound. How can
the whole bushel make a sound, since a large number of no-sounds cannot amount
to any sound.

5. Comments on the Paradoxes: Plurality or Spatial Extension

A thorough discussion of all these paradoxes would take a lot more than can be
fitted in here. The well-known philosopher of science Adolf Grunbaum (1967)
devoted a book to it. Nonetheless, there are a number of interesting principles we
can derive from some of these paradoxes. I begin with Spatial Extension (plural-
ity). The first principle that it depends on is:

**Principle 1**

Any thing with (non-zero) size is divisible into (non-overlapping) parts which are
of smaller (non-zero) size. The size of the whole is the sum of the sizes of those
parts.

This is a natural principle. But we must allow that it might be false, that is that
space might be quantized or granular. By this I mean that space might consist of
atoms of space, which have a least non-zero size, and are indivisible into anything
shorter. This possibility has certainly been countenanced. But notice that the argu-
ment does not depend on things being actually divisible, only being divisible in
thought. Then Zeno's argument can be put through.

Nonetheless, we can sidestep this problem by asking whether as a matter of fact
space is infinitely divisible, or alternatively quantized. It might be thought that the
quantum theory demonstrates that space itself is quantized into minimum non-
zero distances. The answer is no. Without going into technicality, it is only for certain quantum systems that there is quantization of any quantity. For the general case, the answer is no: quantum mechanics has written into it the assumption that space and time are continuous and infinitely divisible. Interested readers may consult Grunbaum (1967:109–114).

Thus, I endorse Principle 1. We can then say that this step in Zeno’s argument is justified. But we can also say that if space were quantized then this step in Zeno’s argument would be blocked, so that motion in quantized space would still be possible. However, we see later, particularly with The Race Track II, that there are other paradoxes that pose problems for quantized space.

The second principle which Zeno’s argument depends on is:

**Principle 2**

Ultimately, division leads to atomic parts (parts that have no proper parts themselves). Atomic parts have no size. That is, all things are composed of things without size.

I endorse this principle also, for similar reasons to those just given for Principle 1. One might argue that if atomic parts had size, they would have proper parts (parts of a lesser size). But in any case, points of space are generally said in both quantum theory and relativity theory to have no size.

The third principle we can learn, and which I will endorse, is one which Zeno must deny.

**Principle 3**

It is possible for something having a (non-zero) size to be composed of an infinite number of things having no size.

As is apparent, it is Zeno’s presupposition that this is false, which leads to the paradox. We can thus learn from the truth of Principle 3. It is allowed by various nineteenth- and twentieth-century mathematical theories of size or distance (metric spaces, measure theory, Riemannian spaces). Perhaps the simplest of these is the theory of *metric spaces*. To sketch it briefly, a metric or distance function is a function \( d(X,Y) \) between all the pairs of points \( X \) and \( Y \) in a space satisfying:

\[
\begin{align*}
(1) & \quad d(X,Y) \geq 0 \\
(2) & \quad d(X,Y) = 0 \text{ iff } X = Y \\
(3) & \quad d(X,Y) + d(Y,Z) \geq d(X,Z)
\end{align*}
\]

This shows that the theory of size or distance postulates that distance is a relation between pairs of points in a space. It is not something that somehow accumulates from the absolute size of single points in the space. It follows that we have every reason to affirm Principle 3, and every reason to reject Zeno’s move in this paradox: modern mathematics affirms Principle 3 also. Thus, as far as Zeno’s argument goes,
motion remains possible. Zeno's argument is erroneous. But it took over two millennia for that to be clarified by modern mathematics.

6. Two Paradoxes of Motion

The first two paradoxes, namely The Race Track I and Achilles and the Tortoise, depend on the premiss that one cannot complete an infinite number of (non-overlapping) tasks in a finite time. Notice that if one did not have to complete an infinite number of tasks, but only a finite number as would be the case if space were quantized, then there would be no paradox. Thus these paradoxes are directed at showing that motion in a continuous infinitely-divisible space is impossible. However, as we have already seen, the assumption that space is continuous is a reasonable one, supported by twentieth-century physics. The significant lesson to be learned from these premisses is therefore different. It is the denial of Zeno's first premiss, namely:

**Principle 4**

It is possible to complete an infinite number of tasks in a finite time.

The simplest way to see that completing an infinite number of distinct tasks in a finite time is possible, is to note that in any motion at all, I traverse an infinite number of positions in space (all the points between starting and stopping). But perhaps that smacks too much of question-begging, since after all Zeno was trying to prove that motion in a continuous space is impossible. A more realistic justification of Principle 4 is familiar to us from modern algebra: infinite sums of non-zero quantities can be finite. For example, it is well-known that $1/2 + 1/4 + 1/8 + ... + 1/2^n + ... = 1$. But the terms on the left hand side exactly parallel the tasks required in our two paradoxes. This shows that the two paradoxes, in relying on the idea that an infinite number of tasks cannot be completed in a finite time, are fallacious.

7. The Race Track II

The third paradox of motion has a different lesson to be learnt. In its simplest terms, it is an attack on quantized or discrete time. Imagine two series of moving bodies passing one another in discrete time. At time $t=1$, point 1 on body 1 is opposite point 1 on body 2. At the next instant $t=2$, point 1 on body 1 is opposite point 3 on body 2 (since both move one unit in opposite directions). So we have:

<table>
<thead>
<tr>
<th>Time = 1</th>
<th>Time = 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Body 1 1, 2, 3 (moving to right)</td>
<td>Body 1 1, 2, 3</td>
</tr>
<tr>
<td>Body 2 1, 2, 3 (moving to left)</td>
<td>Body 2 1, 2, 3</td>
</tr>
</tbody>
</table>
The problem thus is: what happened to the intermediate state where 1 on body 1 is opposite 2 on body 2? There is no intermediate time to fit it onto, because Time=2 is the very next time after Time=1. And yet surely the system must have gone through the intermediate state to get to where it is at Time=2.

The solution recommended by Grunbaum (1967:109–114) can be stated as another Principle:

**Principle 5**

In motion in discrete time, some intermediate descriptions are not realized as events.

In essence, he is saying that the conclusion of the Zeno argument, that apparently possible state-descriptions do not occur in discrete time, is to be accepted. But he stresses, contrary to Zeno, that this is in no way an *a priori* objection to discrete time. That seems right: discrete time does seem to be consistent (albeit empirically false). It is merely that, given the assumption of discrete time, one has to revise one’s view of which things actually occur, i.e. become events.

It is apparent that the paradoxes we have looked at so far are all tricky, and the various lessons to be learnt are subtle ones about space, time and motion. One even more troublesome is The Arrow.

**8. The Arrow**

The Arrow, especially as directed against motion in continuous space and time, really does give one pause. Freeze-frame the universe. *Nothing is moving. Motion does not exist.* How then could any number of non-existent motions add up to an existent motion?

Grunbaum approvingly quotes Russell’s solution (Russell, 1937:351). Motion, like distance, is *relational*. (It is essentially given by the first and second derivatives of distance over time, i.e. the limit of a series of quotients \(\delta x/\delta t\).) Thus, as with Extension (Principle 3), it is a mistake to expect non-zero motion to accumulate from a series of nonexistent motions.

But is it though? Both Russell and Grunbaum note a consequence for continuous motion: *motion is not a state of a body*. That is, motion is not a property that it holds at a time, irregardless of other times. It is, rather, simply a matter of being in different places at different times, that is a relation between different things and times. But is that right?

Graham Priest (1987) argues that it is not right. Priest rejects what he calls the *cinematic view* of Russell. Instead he follows Hegel in postulating that motion is inherently contradictory. The details of this theory are complex, but let it be said that Priest supposes that the mark of motion is that a moving body inconsistently occupies a lozenge of space, where the body is inconsistently both *at* and *not-at* all the points in the lozenge. The Theory of Inconsistency has by now developed
sufficiently to take on board that this theory of motion as at least *possible*. More than that, it is difficult not to feel sympathy with Hegel and Priest. If motion at any instant is *non-existent*, then how could motion occur, if its motion does not exist at every point? For further discussion, see Mortensen (2002).

I do not pretend that this is a settled matter. To the contrary, it is further cause for celebration of Zeno’s penetrating vision, that the puzzle survives unresolved after 2,500 years. For the same reason, I hesitate to erect a further principle to be learned from our discussion of The Arrow. Motion at least involves relations between bodies, positions and times, but it might be more than that. The matter is unresolved at present.

**9. Place and The Millet Seed**

I will be briefer with these two paradoxes. The paradox of Place, which is directed at both continuous and discrete theories of space, looks like it derives an infinite regress from the premisses. (1) Whatever exists, exists at a place (or point in space). (2) Points of space exist. So, (3) points of space exist at points of space. (Place exists at a place, which exists at a place, ...*ad infinitum.*) The simplest solution would seem to be to deny the first premiss, that whatever exists, exists at a place. We could say that this holds for anything *other* than the existing points of space. Lest this move sound a little *ad hoc*, let it be said that the exception is justified because points of space, rather than existing *at* places, constitute the *standard* of place.

On the other hand, The Millet Seed has been underestimated, I think. It poses a significant question in cognitive science: how can threshold effects in consciousness occur? How can a number of not-noticeable differences make up a noticeable difference? It is the same problem as the so-called *colour Sorites* problem. We are all familiar with the experience of seeing three colours such that A is indistinguishable from B and B is indistinguishable from C, but A and C can definitely be distinguished. How is this possible? The Millet Seed is essentially the same, I suggest, but transposed into auditory experiences. somehow a number of non-existent experiences are made into an existent experience.

Now the brute fact that it does occur should reassure us that it is possible. And the way it was just stated suggests that it is a problem for how we distinguish experiences, that is *mental states*. (There is, of course, a more general Sorites paradox, but that is somewhat different, and not due to Zeno but to Eubulides.) Beyond that, I think that there is an interesting issue raised here of what it means to be present to consciousness. There is, of course, the electrical phenomenon of a threshold effect, for example as exhibited by a capacitor. This must have something to do with the physical explanation, but it still does not quite get to the residual worry or how mental states, whose essence it is to be conscious, can have mental aspects which are somehow beneath consciousness. This can be stated as a final principle. First, a definition: to say that a relation R is *transitive* is to say that (for

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any X, Y, and Z) if X bears R to Y and Y bears R to Z, then X bears R to Z. (As an
eexample of a transitive relation R, let R = “taller than”; that is, if X is taller than Y
and Y is taller than Z then X is taller than Z.) Then we can say of The Millet Seed:

**Principle 6**

Indistinguishability is not a transitive relation.

But this is more to pose the problem of how this interesting fact can be so, than it is to solve it.

I do not mean to say that The Millet Seed has consequences which are as profound as The Arrow. I think that it is an interesting problem, to be sure, but one within the theory of consciousness rather than metaphysics. Nonetheless, if Zeno really thought up this clever paradox too, then it is just more tribute to his brilliance and ingenuity. For a further discussion, see Gregory Vlastos (1967).

**10. Conclusion**

We have seen what I believe are the strongest and most sophisticated versions of Zeno’s paradoxes around. We have also seen that we can take away with us several interesting principles, which were not appreciated for more than two millennia. Finally, we have seen that some of the issues remain unresolved. I take it that this constitutes a vindication of Russell’s high opinion of Zeno, as promised in the Introduction.

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