

Control of Singularly Perturbed Hybrid Stochastic Systems

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Abstract

In this paper we study a class of optimal stochastic control problems involving two different time scales. The fast mode of the system is represented by deterministic state equations whereas the slow mode of the system corresponds to a jump disturbance process. Under a fundamental "ergodicity" property for a class of "infinitesimal control systems" associated with the fast mode, we show that there exists a limit problem which provides a good approximation to the optimal control of the perturbed system. Both the finite and infinite discounted horizon cases are considered. We show how an approximate optimal control law can be constructed from the solution of the limit control problem. In the particular case where the infinitesimal control systems possess the so-called turnpike property, i.e. are characterized by the existence of global attractors, the limit control problem can be given an interpretation related to a decomposition approach. Due to the constraints on page numbers all results are presented without proofs. Full details will be supplied in a follow up paper by the same authors.

1 Introduction

This paper deals with the approximation of the optimal control of a class of hybrid *Piecewise Deterministic Control Systems* (PDCS), where the jump disturbances are state and control dependent and when the time scales of the stochastic and the deterministic parts are of different orders of magnitude. More precisely we shall assume that the deterministic state equations defining the evolution of the "continuous" state variable correspond to the *fast mode* of the system whereas

the "discrete" state variable which evolves according to a stochastic jump process defines the *slow mode*.

PDCS are known to provide an elegant paradigm for the study of manufacturing systems (see [17], [7], [2], [18]). Typically, in these models, the stochastic jump process describes the evolution of the operational state of a flexible manufacturing shop, with jumps due to failures and repairs of the machines, whereas the deterministic state equations represent the evolution of the surplus of parts produced by the system. In most of these models the jump Markov disturbances due to failures and repairs are assumed to be represented as a continuous homogenous Markov chain with jump rates which are independent of state and control. In [8] a model has been proposed where, for each machine of the shop, an additional state variable records the *age* of the machine and the failure rates are age dependent. This model provided an example of a PDCS with state dependent jump rates. In [9] a manufacturing system with control (production rate) dependent failure rates is studied.

The class of systems we study in this paper corresponds to a situation where a basically deterministic plant (for example a production system), called the fast subsystem is subject to infrequent modal disruptions occurring randomly (for example the machine failures process), called the slow subsystem. The limit optimal control problem, obtained when the time scale ratio between the slow and the fast processes tends to infinity, is non-trivial as long as the transition probabilities for the perturbing stochastic process depend on the control exercised on the fast system and on its state evolution. In a production system environment this would be the case if, among the (fast) state variables one has, for example, the temperature or the pressure which not only influences the yield of the process but also influences the probability of failures. Indeed, this defines an environment which is natural but significantly different from the one considered in [17].

The method of approximation of the optimal control

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proposed in this paper is related to the theory of control of singularly perturbed systems. A traditional approach to the control of singularly perturbed systems is to equate the perturbation parameter to zero and then use the so-called "Boundary Layer Method". This *reduction technique* proved to be very successful in many applications (see overviews in [5], [14], [15], [16]). We shall use a different approach here. It is related to the *averaging technique* developed in [3], [4], [11]. The technique uses the dynamic programming tenet of transition associated with a change of time scale in a class of locally defined *infinitesimal control problems*. The technique has been mostly used for singularly perturbed deterministic systems and the results reported here seem to be its first adaptation to a stochastic control context. We specialize the analysis to a class of singularly perturbed PDCS's that lend themselves nicely to a nice dynamic programming approach which is well adapted to our averaging technique. In [12] a different averaging technique is proposed for the analysis of singularly perturbed controlled jump-diffusion processes. The very general technique of [12] is based on Martingale theory and weak convergence of probability measures. Our method uses more straightforward analysis to derive the approximation error bounds. More importantly, in our approach we have been able to analyse the case where the fast mode of the system is controlled, which belongs to a notoriously difficult class of problems (cf [12]).

2 A Two-Time-Scale Piecewise Deterministic Control System

We consider a hybrid control system with a "fast" mode described through deterministic state equations and a "slow" mode described as a continuous time stochastic jump process.

2.1 Fast deterministic system

Assume that a "continuous" state variable $x \in \mathbb{R}^p$ is "moving fast" according to the state equation

$$\varepsilon \frac{dx}{dt} = f^i(x, u) \quad (1)$$

$$u \in U^i \quad (2)$$

where $U^i \subset \mathbb{R}^m$ is a given control constraint set and $f^i(x, u)$ satisfies the usual smoothness conditions for optimal control problems (C^1 in x , continuous in u). This state equation is indexed over a finite set ($i \in I$) which describes the different possible operational modes of the system. The perturbation parameter ε will eventually tend to 0. An admissible control for the system (1)-(2) is a measurable function $u(t)$ taking its values in U^i so that the solution of (1) exists and is unique for any initial values from a sufficiently large domain.

It will be convenient to define a "stretched out time scale" via the transformation $\tau = \frac{t}{\varepsilon}$. In this case, given an initial state x^0 and an admissible control $u(t)$ there exists a unique trajectory $\tilde{x}(\cdot) : [0, \infty) \mapsto \mathbb{R}^p$ which is the solution to

$$\frac{d\tilde{x}(\tau)}{d\tau} = f^i(\tilde{x}(\tau), \tilde{u}(\tau)) \quad (3)$$

$$\tilde{u}(\tau) \in U^i \quad (4)$$

$$\tilde{x}(0) = x^0, \quad (5)$$

where we have used the following notations $\tilde{x}(\tau) = x(\varepsilon\tau)$, $\tilde{u}(\tau) = u(\varepsilon\tau)$.

2.2 Slow stochastic jump process

We assume that a discrete state variable is "moving slowly" according to a continuous time stochastic jump process with transition rates (for $i \neq j$)

$$P[\xi(t + \delta) = j | \xi = i, x(t) = x, u(t) = u] = q_{ij}(x, u)\delta + o(\delta) \quad (6)$$

$$\lim_{\delta \rightarrow 0} \frac{o(\delta)}{\delta} = 0, \quad (7)$$

where the $q_{ij}(x, u)$ are continuous functions and limit in (7) is uniform in x and u from a sufficiently large domain. Here, as usual, $q_{ij}(\cdot, \cdot) \geq 0$ if $i \neq j$, $q_{ii}(\cdot, \cdot) < 0$, and $\sum_{j \in I} q_{ij}(\cdot, \cdot) = 0$. We also use the notation $q^i(\cdot, \cdot) = \sum_{j \neq i} q_{ij}(\cdot, \cdot) > 0$ for the jump rate of the ξ -process.

2.3 Admissible policies and performance criterion

Let $L^i(x, u)$ be a continuous function which gives the rate at which cost accumulates in this system. We define a control policy as follows.

Let $s = (\xi, x)$ be the hybrid composite state of the system. Let t_n denote the n -th jump time of the ξ process. At $t_n < T$ the controller observes $s^n = (i^n, x^n) = s(t_n)$ and chooses an admissible control $u(\cdot) : [t_n, T] \mapsto U^{i^n}$. The associated trajectory $x^n(\cdot) : [t_n, T] \mapsto \mathbb{R}^p$ is the solution of

$$\varepsilon \frac{dx^n(t)}{dt} = f^{i^n}(x^n(t), u^n(t)) \quad (8)$$

$$u^n(t) \in U^{i^n} \quad (9)$$

$$x^n(t_n) = x^n. \quad (10)$$

This control and trajectory will be acting until the ξ -process jumps again at time t_{n+1} or until the terminal time T if that is reached first. A policy is a mapping $\gamma(\cdot, \cdot, \cdot) : [0, T] \times I \times \mathbb{R}^p \mapsto \mathcal{U}(t, i)$ where $\mathcal{U}(t, i)$ is the class of measurable mappings $u(\cdot) : [t, T] \mapsto U^i$. A policy defines the action (i.e. the selection of a control) to be chosen at each jump time of the ξ -process. A policy is admissible if it defines a measurable random process $\{t_n, s^n\}$, $n = 0, 1, \dots$

Associated with an initial time \bar{t} , state \bar{s} and an admissible policy γ we define the following performance criterion, for the time interval $[\bar{t}, T]$,

$$J_\varepsilon^\gamma(\bar{t}, \bar{s}) = E_\gamma \left[\int_{\bar{t}}^T L^{\xi(t)}(x(t), u(t)) dt + G(\xi(T)) | s(\bar{t}) = \bar{s} \right], \quad (11)$$

where $G(i)$ is a terminal cost incurred when $\xi(T) = i$. Notice that we assume that this terminal cost does not depend on the value of the "fast" state variable $x(T)$. We are interested in approximating the optimal value function

$$J_\varepsilon^*(t, s) = \inf_\gamma J_\varepsilon^\gamma(t, s) \quad (12)$$

by a suitably constructed limit value function.

3 Convergence to a Limit-Control Problem

For $\delta > 0$ a policy γ is said to be δ -optimal if $J_\varepsilon^\gamma(\bar{t}, \bar{s}) < J_\varepsilon^*(\bar{t}, \bar{s}) + \delta$ for all \bar{t}, \bar{s} . We make the following coercivity assumption.

Assumption 1 *The class of controls and initial states considered is such that, for some $\delta > 0$, for any δ -optimal policy, $x(t)$ and $u(t)$ remain uniformly bounded over $[0, T]$. From now on we assume that $x(t) \in X$, a bounded subset of \mathbb{R}^p .*

3.1 An associated class of infinitesimal control problems

For any vector $\mathbf{v} = \{v(j)\}_{j \in I}$ consider the family of optimal control problems

$$H^i(\theta, x^0, \mathbf{v}) = \inf \frac{1}{\theta} \int_0^\theta \left(L^i(\tilde{x}(\tau), \tilde{u}(\tau)) + \sum_{j \in I} q_{ij}(\tilde{x}(\tau), \tilde{u}(\tau))v(j) \right) d\tau \quad (13)$$

s.t.

$$\frac{d\tilde{x}(\tau)}{d\tau} = f^i(\tilde{x}(\tau), \tilde{u}(\tau)) \quad (14)$$

$$\tilde{u}(\tau) \in U^i \quad (15)$$

$$\tilde{x}(0) = x^0. \quad (16)$$

These problems, defined over the stretched out time scale, will be called the *infinitesimal control problems*¹. The term *infinitesimal* emphasizes the fact that, in the fast time scale, we shall have an essentially infinite horizon control problem defined locally for almost every intermediate time $\bar{t} \in [0, T]$.

¹The term infinitesimal control problem has been coined by Zvi Artstein.

Assumption 2 *There exist two constants $A > 0$ and $0 < \alpha \leq 1$, and for each i a function $H^i(\mathbf{v})$, such that for all $i \in I$, $x^0 \in X$ and \mathbf{v} in some bounded set V*

$$|H^i(\theta, x^0, \mathbf{v}) - H^i(\mathbf{v})| \leq \frac{A}{\theta^\alpha}. \quad (17)$$

Remark 1 *The above assumption clearly resembles an ergodicity property. When the time horizon θ goes to ∞ , the optimal value becomes independent of the initial state. Such a property is expected if the system admits an optimal steady state which is a common attractor for all optimal trajectories. This has been called the turn-pike property and Section 7 will provide more details concerning this case. Another possibility for observing such an ergodic behavior is to obtain a periodic control when $\theta \rightarrow \infty$.*

Remark 2 *As $H^i(\theta, x^0, \mathbf{v})$ is the value function of a control problem depending on the parameter \mathbf{v} in a linear way, and the bound in (17) is uniform, the "limit" $H^i(\mathbf{v})$ of $H^i(\theta, x^0, \mathbf{v})$ is a Lipschitz function.*

3.2 The limit value function

Consider the set of coupled differential equations

$$\frac{dJ_0(t, i)}{dt} = -H^i(J_0(t)), \quad i \in I, \quad (18)$$

with terminal conditions

$$J_0(T, i) = G(i) \quad i \in I, \quad (19)$$

where we have denoted by $\mathbf{J}_0(t)$ the vector $\{J_0(t, j)\}_{j \in I}$.

Assumption 3 *The system (18), (19) admits a solution which satisfies*

$$\mathbf{J}_0(t) \in V \quad \forall t \in [0, T].$$

Theorem 1 *There exists a constant C such that*

$$|J_\varepsilon^*(t, i, x) - J_0(t, i)| \leq C \varepsilon^{\alpha/(1+\alpha)} \quad \forall i \in I, t \in [0, T], x \in X. \quad (20)$$

4 Approximate Optimal Control

In this section we again use the averaging technique to show that, once the limit problem is solved, it is possible to construct from its solution an approximate control of the perturbed problem.

4.1 Approximate feedback optimal control for the associated control problems

Assumption 4 *There exists, for each $\mathbf{v} \in V$ and $i \in I$, an admissible feedback control denoted $\tilde{u}_{\mathbf{v}}^i(x, \tau) \in U^i$ such that*

$$\left| \frac{1}{\theta} \int_0^\theta h_{\mathbf{v}}^i(\tilde{x}(\tau), \tilde{u}_{\mathbf{v}}^i(\tilde{x}(\tau), \tau)) d\tau - H^i(\theta, x^0, \mathbf{v}) \right| \leq \frac{A}{\theta^\alpha} \quad (21)$$

where V , A and α are as in Assumption 2, $\tilde{x}(\tau)$ is the solution of

$$\frac{d\tilde{x}(\tau)}{d\tau} = f^i(\tilde{x}(\tau), \tilde{u}_{\mathbf{v}}^i(\tilde{x}(\tau), \tau)) \quad (22)$$

$$\tilde{u}_{\mathbf{v}}^i(\tilde{x}(\tau), \tau) \in U^i \quad (23)$$

$$\tilde{x}(0) = x^0, \quad (24)$$

and where we have used the following notation

$$h_{\mathbf{v}}^i(x, u) = L^i(x, u) + \sum_{j \in I} q_{ij}(x, u)v(j). \quad (25)$$

4.2 Control implementation

Let t_ℓ be defined as in the proof of Theorem 1, with $t_0 = 0$, and $t_\ell = \ell \Delta(\varepsilon)$, $\ell = 0, 1, \dots, \lceil \frac{T}{\Delta(\varepsilon)} \rceil = L(\varepsilon)$. On each subinterval $[t_\ell, t_{\ell+1})$ the feedback implemented will be

$$\hat{u}_\ell(\xi, x, t) = \tilde{u}_{\mathbf{v}_{\ell+1}^0}^\xi(x, \frac{t - t_\ell}{\varepsilon}) \in U^\xi, t \in [t_\ell, t_{\ell+1}), \\ x \in X, \xi \in I, \quad (26)$$

where $\mathbf{v}_{\ell+1}^0 = \mathbf{J}_0(t_{\ell+1})$. We shall denote by $J_\varepsilon^{\hat{u}}(0, i, x)$ the expected cost associated with the use of the above defined feedback law, with initial conditions $x(0) = x$, $\xi(0) = i$.

Remark 3 *Notice that such feedbacks will give rise to an admissible policy for the PDCS, in the sense of subsection 2.3, since the representation of the control, along the deterministic sections of the trajectories (i.e. between two successive random jump times) will be an admissible open-loop control.*

4.3 Approximation of the optimal value function

Theorem 2 *Under Assumptions 1-4 the following inequality holds*

$$|J_\varepsilon^{\hat{u}}(0, i, x) - J_\varepsilon^*(0, i, x)| \leq C \varepsilon^{\frac{\alpha}{1+\alpha}}. \quad (27)$$

5 Infinite Horizon with Discounted Cost

In the next three sections we extend the analysis to the case of an infinite horizon control process with discounted integral cost.

5.1 The infinite horizon control problem

We consider the same system as in section 2, with a terminal time $T \rightarrow \infty$. A control policy γ is still defined as in section 2.3, with the obvious replacement of T with ∞ . As usual, when dealing with infinite horizon stationary systems, one may restrict the analysis to a class of stationary policies.

5.2 Performance criterion

Associated with an admissible policy, we define the following performance criterion

$$J_\varepsilon^\gamma(\bar{t}, \bar{s}) = E_\gamma \left[\int_{\bar{t}}^\infty e^{-\rho t} L^\xi(t)(x(t), u(t)) dt | s(\bar{t}) = \bar{s} \right], \quad (28)$$

where $\rho > 0$ is a given discount rate. We are interested in the optimal value function

$$J_\varepsilon^*(t, s) = \inf_\gamma J_\varepsilon^\gamma(t, s). \quad (29)$$

As usual when dealing with discounted cost criterion we shall use the *current-value* cost-to-go value function

$$V_\varepsilon^*(s) = J_\varepsilon^*(0, s) = e^{\rho t} J_\varepsilon^*(t, s). \quad (30)$$

6 Convergence to a Limit-Control Problem

In this section we obtain a convergence result, similar to the one established in Theorem 1 in section 3, but valid for infinite horizon, discounted cost problems.

Let $\mathbf{V}_0 = \{V_0(j)\}_{j \in I}$ be a solution to the algebraic equations

$$\rho V_0(i) = H^i(\mathbf{V}_0) \quad i \in I. \quad (31)$$

Theorem 3 *There exists a constant C such that*

$$|V_\varepsilon^*(i, x) - V_0(i)| \leq C \varepsilon^{\alpha/(1+\alpha)} \quad \forall i \in I, x \in X. \quad (32)$$

7 Approximate Stationary Optimal Feedback Controls

When the system is controlled over an infinite time horizon with stationary state equations and a discounted cost, one can approximate an optimal control policy for the perturbed system via appropriately defined stationary feedbacks.

Assume the following slightly more restrictive version of Assumption 4

Assumption 5 *There exists, for each $\mathbf{v} \in V$ and $i \in I$, an admissible stationary feedback control denoted*

$\tilde{u}_v^i(x) \in U^i$ such that

$$\left| \frac{1}{\theta} \int_0^\theta h_v^i(\tilde{x}(\tau), \tilde{u}_v^i(\tilde{x}(\tau))) d\tau - H^i(\theta, x^0, v) \right| \leq \frac{A}{\theta^\alpha} \quad (33)$$

where A and α are as in Assumption 2, and $\tilde{x}(\tau)$ is solution of

$$\frac{d\tilde{x}(\tau)}{d\tau} = f^i(\tilde{x}(\tau), \tilde{u}_v(\tilde{x}(\tau))) \quad (34)$$

$$\tilde{u}_v^i(\tilde{x}(\tau)) \in U^i \quad (35)$$

$$\tilde{x}(0) = x^0. \quad (36)$$

Given the value vector V_0 obtained from the solution of the limit control problem, we implement the following stationary feedback

$$\hat{u}(\xi, x) = \tilde{u}_{V_0}^\xi(x), x \in X, \xi \in I, \quad (37)$$

Let's call $V_\varepsilon^{\hat{u}}(i, x)$ the discounted expected cost associated with the use of the above defined feedback law, with initial conditions $x(0) = x, \xi(0) = i$.

Theorem 4 Under Assumptions 1-3 and 5 the following inequality holds

$$|V_\varepsilon^{\hat{u}}(i, x) - V_\varepsilon^*(i, x)| \leq C \varepsilon^{\frac{\alpha}{1+\alpha}}. \quad (38)$$

8 Turnpikes and Decomposition Principle for Stationary Convex Systems

In this section we show that the limit-control problem defined in sections 3 and 5 can be easily solved when the system is convex, and that the associated control problems (13) - (16) satisfy the following weak *turnpike* property:

Assumption 6 For each $i \in I$ and each v there exists a unique optimal steady state $\bar{x}_v^i \in X$ with a control $\bar{u}_v^i \in U^i$ such that

$$L^i(\bar{x}_v^i, \bar{u}_v^i) + \sum_{j \in I} q_{ij}(\bar{x}_v^i, \bar{u}_v^i)v(j) =$$

$$\max_{x \in X, u \in U^i} \left\{ L^i(x, u) + \sum_{j \in I} q_{ij}(x, u)v(j) \right\} \quad (39)$$

s.t.

$$0 = f^i(x, u) \quad (40)$$

$$u \in U^i. \quad (41)$$

Furthermore, the following equality holds

$$H^i(v) = L^i(\bar{x}_v^i, \bar{u}_v^i) + \sum_{j \in I} q_{ij}(\bar{x}_v^i, \bar{u}_v^i)v(j). \quad (42)$$

The name *turnpike* has been coined by economists when they applied the optimal control formalism to the optimal economic growth problems (see [19]). For a review of the conditions under which such a property holds we refer to the book [10]. It suffices to say that this property will hold in our context under the following natural assumption:

Assumption 7 The controlled system is such that

1. For each $i \in I$ and $v \in V$ the function $f^i(x, u)$ is linear in x and u , the control set U^i is compact and convex, the function $L^i(x, u) + \sum_{j \in I} q_{ij}(x, u)v(j)$ is strictly convex in x , convex in u ;
2. The set $\bar{X}^i = \{x \in X : 0 = f^i(x, u), u \in U^i\}$ is nonempty for each $i \in I$;
3. Any $\bar{x} \in \bar{X}^i$ can be reached in a uniformly bounded finite time from any initial state x^0 , when the system is in mode $i \in I$.

It will be convenient to introduce the following "action sets" $\mathcal{A}^i = \{\bar{a} = (\bar{x}, \bar{u}) \in X \times U^i : 0 = f^i(\bar{x}, \bar{u})\}$ for each $i \in I$. We then define an *upper level* controlled Markov chain as follows:

- state set I ,
- action set \mathcal{A}^i for each state $i \in I$,
- cost rate $\mathcal{L}^i(\bar{a}) = L^i(\bar{x}, \bar{u})$, when in state $i \in I$ and action $\bar{a} \in \mathcal{A}^i$.
- transition rates $\mathcal{Q}_{ij}(\bar{a}) = q_{ij}(\bar{x}, \bar{u})$.

Then the coupled differential equations (18)-(19) defining the *limit value function* in the finite time horizon case, as well as the algebraic equations (31) in the infinite horizon discounted cost case, correspond exactly to the dynamic programming equations for the *upper level* controlled Markov chain. This permits us to give, in the infinite horizon case, the following interpretation of the limit control problem as a decomposition scheme for the perturbed stochastic control problem:

Let $m = |I|$. Consider a set of $m + 1$ agents controlling the system. Each agent $i = 1, \dots, m$ controls the fast system when the discrete mode is $i \in I$. Agent 0 is a *coordinator*. The coordinator solves the *upper level* controlled Markov chain problem and sends to each agent $i = 1, \dots, m$ the optimal limit value vector $V_0 = \{V_0(j) : j \in I\}$.

Now, given this information, agent i constructs an auxiliary cost rate

$$h_{V_0}^i(x, u) = L^i(x, u) + \sum_{j \in I} q_{ij}(x, u) V_0(j)$$

and pilots the system, when it is in operational mode i , as if it were a deterministic control problem, with an infinite time horizon and an average cost criterion. As soon as the system jumps to state k , agent k constructs $h_{V_0}^k(x, u)$ and proceeds in similar manner, and so on.

In the finite horizon case a similar, although more involved, interpretation could be developed:

With the same setting of $m + 1$ agents as above, the coordinator will send an information in the form of a limit value function $V_0(t) = \{V_0(t, j) : j \in I\}$, $t \in [0, T]$, obtained from the solution of the upper level controlled Markov chain problem on the time horizon $[0, T]$. Then, at each instant $\bar{t} \in [0, t]$ the agent $i \in I$ would have to solve an infinitesimal control problem which, in the stretched out time scale would also correspond to an infinite horizon deterministic control problem with cost rate

$$L^i(x, u) + \sum_{j \in I} q_{ij}(x, u) J_0(\bar{t}, j).$$

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