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Local uniform convexity and Kadec-Klee type properties in K-interpolation spaces I : General Theory

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Abstract. We present a systematic study of the interpolation of local uniform convexity and Kadec-Klee type properties in K-interpolation spaces. Using properties of the K-functional of J.Peetre, our approach is based on a detailed analysis of properties of a Banach couple and properties of a K-interpolation functional which guarantee that a given K-interpolation space is locally uniformly convex, or has a Kadec-Klee property. A central motivation for our study lies in the observation that classical renorming theorems of Kadec and of Davis, Ghoussoub and Lindenstrauss have an interpolation nature. As a particular by-product of our study, we show that the theorem of Kadec itself, that each separable Banach space admits an equivalent locally uniformly convex norm, follows directly from our approach.

1. Introduction

A Banach space E is said to have the Kadec-Klee property (sometimes called the Radon-Riesz property, or property (H)) if weak and norm

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convergence of sequences coincide on the unit sphere. Closely related to the Kadec-Klee property is that of local uniform convexity, sometimes referred to as local uniform rotundity or the (LUR)-property: $x, x_n \in E, n = 1, 2, \ldots, \|x_n\|_E \to \|x\|_E$ and $\|x + x_n\|_E \to 2\|x\|_E$ imply that $x_n \to x$ in the norm of E. These properties belong to a large family of geometric properties which arise in the study of the smoothness and rotundity of the unit sphere in normed spaces and have been widely and systematically studied in the literature, and are of interest not only in their own right but also from the point of view of applications. See, for example, [1], [6], [14], [15], [28], [29] and the references contained therein.

This paper studies local uniform convexity and Kadec-Klee type properties in spaces obtained by the K-method of interpolation. The question as to whether a given geometrical property passes from one or both the spaces of a Banach couple $\mathbf{E} = (E_0, E_1)$ to the interpolation space $\mathcal{F}(\mathbf{E})$ has been considered in the literature for various interpolation functors $\mathcal{F}$. For example, in the case of the Lions-Peetre interpolation spaces, Beauzamy [3] showed that if one of the spaces E_0, E_1 is uniformly convex, then so is the corresponding interpolation space. See also [24], where a number of related results are cited. Many results of similar type are known for the complex method of interpolation. Finally, we mention the series of papers [1], [2], [7], [9], [11] which systematically study the "lifting" of local uniform convexity and various Kadec-Klee properties from a given symmetric Banach function space on the positive semi-axis to the corresponding symmetric space of measurable operators affiliated with some semifinite von Neumann algebra. Such spaces may be viewed as K-interpolation spaces.

Our approach is based on a detailed analysis of properties of a Banach couple $\mathbf{E}$ and properties of a K-interpolation functional Φ which guarantee that a K-interpolation space $\mathbf{E}_\Phi$ is locally uniformly convex or has some Kadec-Klee property. Our analysis shows that both the Banach couple itself and the K-interpolation functional should possess "good" properties and the second and third sections consider these properties in detail. The general approach of the present paper brings a great deal of insight into and unifies many results in the study of Kadec-Klee properties and local uniform convexity from a number of sources, including [8],[9],[27],[28]. A principal motivation is to be found in the classical result of Kadec [18], [19], [20] (see also [14]) that every separable Banach space admits an equivalent locally uniformly convex norm. Several authors have given refinements of the Kadec theorem for special classes of Banach spaces with some additional structural properties. For example, it was shown by Sedaev [27], [28] that any separable symmetric (rearrangement invariant) space on the positive half-line admits an equivalent Kadec-Klee norm which is symmetric. In a similar vein, Davis, Ghoussoub and Lindenstrauss [14], [15] showed that each order continuous Banach lattice admits an equivalent locally uniformly convex norm which is a lattice norm. In the present paper a central role is played by a key property of a general Banach couple $\mathbf{E}$, which we call here the (DGL)-property. This property first appeared implicitly in [14] in the special case of the Banach couple $(L_1, L_\infty)(\Omega, P)$, with (Ω, P) a probability space and was used by them in a crucial way to prove their renorming theorem. See also [9] where a similar property arose in the study of local uniform convexity in the setting of non-commutative spaces of measurable operators.

We introduce the (DGL)-property for a given Banach couple in Section 2 below, together with a related property which we call property $(D_{\mathcal{T}})$ which plays a role in our discussion of Kadec-Klee properties for various linear topologies $\mathcal{T}$ similar to that played by the (DGL)-property in the case of local uniform convexity. The principal results of this section (Theorems 3.6, 3.8) show that, subject to mild restrictions on the spaces E_0, E_1, the Banach couple $\mathbf{E} = (E_0, E_1)$ automatically possesses the (DGL)-property (respectively, the property $D_{\mathcal{T}}$) whenever either or both of the spaces E_0, E_1 is locally uniformly convex (respectively, has the Kadec-Klee property with respect to $\mathcal{T}$). We illustrate the effectiveness of our approach based on the (DGL)-property by giving a new proof of the classical renorming theorem of Kadec (Corollary 3.12).

In the third section, we consider a general K-interpolation functional defined on a cone $\mathcal{Q}$ which contains the cone of non-negative concave functions on the semi-axis. We introduce a number of properties that such a functional may possess which find their origin in properties which arise in the study of local uniform convexity and Kadec-Klee properties in the setting of symmetric Banach function spaces [8]. A principal result of this section (Theorems 4.2) shows that if a Banach couple has the (DGL)-property and if the K-interpolation functional Φ satisfies natural rotundity and continuity conditions then the corresponding K-interpolation space is necessarily locally uniformly convex. A similar result holds for Kadec-Klee properties (Theorem 4.1).

Two basic methods of constructing K-interpolation functionals are considered in the fourth section. These functionals yield as special cases the well-known Lions-Peetre spaces $\mathbf{E}_{\theta,p}$ [4], the discrete K-interpolation spaces as given, for example, in [24] and the non-commutative Banach spaces of measurable operators studied in [7], [9], [11]. These functionals depend in a natural way on certain Banach function spaces F, G which serve as parameter spaces. We first show (Theorems 5.1, 5.2) how rotundity and continuity properties of the K-interpolation functional may be inferred from corresponding properties of the parameter spaces. We then systematically refine the general results of Section 3 for arbitrary

K-interpolation functionals into more explicit criteria appropriate to the particular class of K-functionals under consideration, formulated in terms of the (DGL) or $(D_{\mathcal{T}})$-properties of the Banach couple, and readily verifiable geometric properties of the parameter space underlying K-method. As a simple and immediate consequence in the case of the discrete K-method, we show (Corollary 5.9) that every locally uniformly convex Banach space with an unconditional basis is isomorphic to a complemented subspace with a 1-symmetric basis. This complements a similar result for uniformly convex Banach spaces with an unconditional basis given in [23] 3.b.2.

Some of the results of this paper have been announced previously in [29].

2. Preliminaries

We begin by recalling some basic terminology from the theory of Banach function spaces (ideal lattices). In the sequel, $(F, \|\cdot\|_F)$ will always denote a Banach function space on the measure space (T, μ) where T is either the interval $[0, l)$ for some $0 < l \leq \infty$ and μ is Lebesgue measure m, or T is a countable subset $\{t_n\}_{n\in\mathbb{N}} \subseteq \mathbb{R}^+$ and μ is counting measure. Following [21] (see also [4]), a Banach function space (or ideal lattice) on (T, μ) is a Banach space $(F, \|\cdot\|_F)$ which is a linear subspace of the space $L^0(T, \mu)$ of all μ-measurable, almost everywhere finite, complex-valued functions on T which has the property that

$$f \in F,\ g \in L^0(T,\mu),\ |g| \leq |f| \implies g \in F \text{ and } \|g\|_F \leq \|f\|_F.$$

Unless stated otherwise, we shall always assume that the norm $\|\cdot\|_F$ on the Banach function space F is a *Fatou norm*, that is, $\|\cdot\|_F$ is lower semicontinuous on F for the topology $(l\mu)$ of local convergence in measure (that is, convergence in measure on every set of finite measure), in the sense that

$$f, f_n \in F, \quad n = 1, 2, \ldots, \quad f_n \to_{l\mu} f \implies \|f\|_F \leq \underline{\lim}_n \|f_n\|_F.$$

Let us note that the norm $\|\cdot\|_F$ is a Fatou norm if and only if the unit ball of F is closed with respect to (sequential) local convergence in measure, and in turn this is equivalent to the condition

$$f, f_n \in F, \quad n = 1, 2, \ldots, \quad f_n \uparrow f \implies \|f_n\|_F \uparrow \|f\|_F.$$

The Banach function space F is said to have the *Fatou property* if whenever $f \in L^0(T, \mu), f_n \in F, \quad n = 1, 2, \ldots,$ satisfy $f_n \to_{l\mu} f$ and $\underline{\lim}_n \|f_n\|_F < \infty$, it follows that $f \in F$ and $\|f\|_F \leq \underline{\lim}_n \|f_n\|_F$.

The norm on F is said to be *order continuous* if and only if whenever $f_n \in F,\ n = 1, 2, \ldots,$ satisfies $|f_n| \downarrow_n 0$, it follows that $\|f_n\|_F \to 0$. We remark that if F has order continuous norm, then $\|\cdot\|_F$ is a Fatou norm.

If $f \in L^0(T, \mu)$, the *decreasing rearrangement* f^* of f is defined by setting

$$f^*(t) \ = \ \inf\{s : \mu\{|f| > s\} < t\},$$

for all $t > 0$. The rearrangement f^* is right-continuous, decreasing and is equimeasurable with $|f|$ in the sense that

$$m\{f^* > t\} \ \leq \mu\{|f| > t\},$$

for every $t \in \mathbb{R}^+$. The Banach function space F on (T, μ) is said to be *rearrangement-invariant* (or symmetric in the terminology of [21]) if and only if, whenever $g \in F$ and $f^* \leq g^*$, then $f \in F$ and $\|f\|_F \leq \|g\|_F$. We note that the familiar Orlicz, Lorentz and Marcinkiewicz spaces are rearrangement-invariant. For further details, we refer to [21], [4]. We note, in particular, that the norm on a symmetric Banach function space F on $[0, \alpha),\ 0 < \alpha \leq \infty$, with respect to Lebesgue measure m is order continuous if and only if F is separable.

If $f, g \in L^0(T, \mu)$ are locally integrable, we will write $f \prec g$ if and only if

$$\int_0^t f^*(s)ds \leq \int_0^t g^*(s)ds,$$

for all $t > 0$. If F is a rearrangement-invariant Banach function space on (T, μ), then we will say that F is *fully symmetric* if and only if $f \prec g,\ g \in F$ imply $f \in F$ and $\|f\|_F \leq \|g\|_F$. It is a well-known result of A.P. Calderon ([21], [4]) that F is an interpolation space for the Banach couple (L^1, L^∞) with interpolation constant 1 if and only if F is fully symmetric. We shall defer further discussion of interpolation properties of Banach couples to later sections.

We now gather some basic definitions and properties related to the metric geometry of normed linear spaces.

We recall that a Banach space X is said to be *uniformly convex* if and only if for all sequences $\{x_n\}_{n=1}^\infty, \{y_n\}_{n=1}^\infty \subseteq X$

$$\|x_n\|_X, \|y_n\|_X \to d, \|x_n + y_n\|_X \to 2d \Longrightarrow \|x_n - y_n\|_X \to 0.$$

If X is a Banach lattice, then X is said to have the *uniform strong majorant property* if and only if for all sequences $\{x_n\}_{n=1}^\infty, \{y_n\}_{n=1}^\infty \subseteq X$ such that $0 \leq x_n \leq y_n, n = 1, 2, \ldots$ and $\|x_n\|_X, \|y_n\|_X \to d$, it follows that $\|x_n - y_n\|_X \to 0$. Let us note that if the Banach lattice X is uniformly convex, then X has the uniform strong majorant property. In fact, if $\{x_n\}_{n=1}^{\infty}, \{y_n\}_{n=1}^{\infty} \subseteq X$ satisfy $0 \leq x_n \leq y_n, n = 1, 2, \ldots$ and $\|x_n\|_X, \|y_n\|_X \to d$, then it follows from the inequalities

$$2\|x_n\|_X \leq \|x_n + y_n\|_X \leq 2\|y_n\|_X$$

that $\|x_n + y_n\|_X \to 2d$. Uniform convexity of X now implies that $\|x_n - y_n\|_X \to 0$ and consequently X has the uniform strong majorant property.

A Banach space $(E, \|\cdot\|)$ is said to be *locally uniformly convex* if and only if whenever $x_n, x \in E, n = 1, 2, \ldots, \|x_n\| \to \|x\|$ and $\|x_n + x\| \to 2\|x\|$ it follows that $x_n \to_E x$. We shall need the following reformulation in terms of an appropriate modulus of convexity (see, for example [10], Chapter 7).

For every $x \in E$ with $\|x\| = 1$, the function δ_x is defined by setting

$$\delta_x(t) \ = \ \inf\left\{1 - \left\|\frac{x+y}{2}\right\| \ : y \in E,\ \|y\| = 1, \|x - y\| \geq t\right\},$$

for $t \in [0, 2]$. It follows that δ_x is increasing on $[0, 2]$. Moreover, if $0 \neq x, y \in E$, then

$$\|x + y\| \ \leq \|x\| + \|y\| - \delta_{\overline{x}}\left(\left\|\frac{\overline{x} - \overline{y}}{2}\right\|\right) \cdot \min(\|x\|, \|y\|),$$

where $\overline{z} = z/\|z\|$ for all $0 \neq z \in E$. It is now a simple matter to verify the following characterisation of local uniform convexity.

Proposition 2.1 *The Banach space* $(E, \|\cdot\|)$ *is locally uniformly convex if and only if* $\delta_{\overline{x}}(t) > 0$ *for all* $0 \neq x \in E$ *and for all* $t \in (0, 2]$.

We remark that if $(E, \|\cdot\|)$ is locally uniformly convex, if $0 \neq x \in E$, and if $\{t_n\}_{n=1}^{\infty} \subseteq (0, 2]$ is such that $\delta_{\overline{x}}(t_n) \to_n 0$, then it follows that $t_n \to 0$.

We now suppose that $\mathcal{T}$ is a Hausdorff linear topology on the Banach space $(E, \|\cdot\|)$ that is weaker than the norm topology. We denote by $x_n \to_{\mathcal{T}} x$ the convergence of the sequence $\{x_n\} \subseteq E$ to $x \in E$ with respect to $\mathcal{T}$. The norm on E is said to be *$\mathcal{T}$-lower semicontinuous* if $x_n, x \in E, n = 1, 2, \ldots$, and $x_n \to_{\tau} x$ imply

$$\|x\| \leq \underline{\lim}_n \|x_n\|.$$

We remark that the norm on E is $\mathcal{T}$-lower semicontinuous if and only if the unit ball in E is sequentially $\mathcal{T}$-closed.

The Banach space $(E, \|\cdot\|)$ is said to have the *Kadec-Klee property* with respect to the topology $\mathcal{T}$ if $x_n, x \in E, n = 1, 2, \ldots, x_n \to_{\mathcal{T}} x$ and $\|x_n\| \to \|x\|$ implies that $x_n \to_E x$. The result which follows is well-known. See, for example, [6]. We include a proof for sake of completeness.

Proposition 2.2. *If E has the Kadec-Klee property with respect to $\mathcal{T}$ then the norm on E is $\mathcal{T}$-lower semicontinuous.*

Proof. If the norm on E is not $\mathcal{T}$-lower semicontinuous, then there exist $x_n, x \in E, n = 1, 2, \dots,$ and $\epsilon > 0$ such that $x_n \to_{\mathcal{T}} x$, $\lim \|x_n\|$ exists and

$$\|x_n\| < \|x\| - \epsilon, \tag{2.1}$$

for all n. Observe that

$$\|(\alpha+1)x_n - \alpha x\| \ \leq \ \|x_n\| + \alpha\|x_n - x\| \ < \ \|x\|,$$

for all sufficiently small $\alpha > 0$ for all $n = 1, 2, \dots,$ and that

$$\|(\alpha+1)x_n - \alpha x\| \ \geq \ \alpha\|x\| - (\alpha+1)\|x_n\| \ = \ \alpha(\|x\| - \|x_n\|) - \|x_n\| \ > \ \|x\|,$$

for all sufficiently large $\alpha > 0$ for all $n = 1, 2, \dots$. Accordingly, there exist numbers a, b such that $0 < a < b < \infty$ and for each $n = 1, 2, \dots,$ there exists $\alpha_n \in [a, b]$ such that

$$\|(\alpha_n + 1)x_n - \alpha_n x\| \ = \ \|x\|.$$

Passing to a subsequence if necessary we may assume that $\alpha_n \to \alpha_0$. It is clear that

$$(\alpha_0 + 1)x_n - \alpha_0 x \ \longrightarrow_{\mathcal{T}} \ x, \tag{2.2}$$

and

$$\|(\alpha_0 + 1)x_n - \alpha_0 x\| \ \longrightarrow \ \|x\|. \tag{2.3}$$

Since E has the Kadec-Klee property with respect to $\mathcal{T}$, it follows from (2.2) and (2.3) that $\|x_n - x\| \to 0$ which contradicts (2.1). □

Proposition 2.3. *Let $\mathcal{T}$ be a Hausdorff linear topology on the Banach space $(E, \|\cdot\|)$ that is weaker than the norm topology. If E is locally uniformly convex and if the norm on E is $\mathcal{T}$-lower semicontinuous, then E has the Kadec-Klee property with respect to $\mathcal{T}$.*

Proof. Let $x_n, x \in E,\ n = 1, 2, \dots,$ satisfy $x_n \to_{\mathcal{T}} x$ and $\|x_n\| \to \|x\|$. Then $x_n + x \to_{\mathcal{T}} 2x$ and by the assumption that the norm on E is $\mathcal{T}$-lower semicontinuous, it follows that

$$\|2x\| \ \leq \ \underline{\lim}_n \|x_n + x\| \ \leq \ \overline{\lim}_n \|x_n + x\| \ \leq \|x\| + \lim_n \|x_n\| = \ 2\|x\|.$$

Since E is locally uniformly convex, it now follows that $x_n \to_E x$. □

We obtain immediately the following consequence.

Corollary 2.4. *Let E be a Banach space and let $\mathcal{T}$ be either the weak topology or the weak* topology on E, in the case that E is a dual space. If E is locally uniformly convex, then E has the Kadec-Klee property with respect to $\mathcal{T}$.*

Before proceeding, we recall (see [8]) that the norm $\|\cdot\|_F$ on the Banach function space F is said to be *strictly monotone* if and only if

$$x, y \in F, \quad 0 \le x \le y, \quad x \ne y \Rightarrow \|x\|_F < \|y\|_F.$$

The norm $\|\cdot\|_E$ on the symmetric Banach function space E is said to be *strictly K-monotone* if and only if

$$x, y \in E, \quad x \prec y, \quad x^* \ne y^* \Rightarrow \|x\|_E < \|y\|_E.$$

Corollary 2.5 *Let $(F, \|\cdot\|_F)$ be a Banach function space on (T, μ). If F is locally uniformly convex, then the norm on F is strictly monotone and F has the Kadec-Klee property for the topology $(l\mu)$ on F of local convergence in measure.*

Proof. It is easily verified that if F is locally uniformly convex, then the norm on F is strictly monotone. By [14] Theorem 1.2, local uniform convexity of F implies that the norm on F is order continuous. Consequently, the norm on F is a Fatou norm and hence is lower semicontinuous for the topology of local convergence in measure. The second assertion now follows immediately from Proposition 2.3. □

We shall also need the following result which is proved as in [8], Proposition 1.1.

Proposition 2.6. *Let $(F, \|\cdot\|_F)$ be a Banach function space on (T, μ). If $\mathcal{T}$ is the weak topology on F, or the weak* topology if F is a dual space, or is the topology of local convergence in measure, and if F has the Kadec-Klee property with respect to $\mathcal{T}$, then the norm on F is order continuous.*

The relationship of strictly monotone and strictly K-monotone norms to various Kadec-Klee properties in symmetric function spaces is studied in some detail in [8]. For example, it is shown there (Theorem 4.2) that separable symmetric spaces on $[0, \infty)$ with the Kadec-Klee property for local convergence in measure necessarily have strictly monotone norms, although this is not the case for symmetric spaces on the interval $[0, 1)$. For separable Lorentz spaces, the Kadec-Klee property (for the weak topology) is in fact equivalent to strict K-monotonicity of the norm ([8], Theorem 2.10). Together with the results cited above, this shows that local uniform convexity and Kadec-Klee properties necessarily imply a number of weaker function space properties such as strict monotonicity, order continuity and lower semi-continuity of the norm. In subsequent sections, we shall give conditions in terms of these properties on a Banach couple $\mathbf{E}$ and a K-interpolation functional Φ that will suffice for the norm on the K-interpolation space $\mathbf{E}_\Phi$ to be locally uniformly convex or Kadec-Klee.

We denote by Q the set of all nonnegative increasing concave functions on $\mathbb{R}^+ = (0,\infty)$ and by Q_0 the set $\{f \in Q : f(+0) = 0\}$. It is clear that Q and Q_0 are cones which are closed under dilation, and under pointwise convergence. Moreover, the pointwise infimum of any family of elements of Q (respectively Q_0) is again an element of Q (respectively Q_0). Each element $f \in Q$ is absolutely continuous and admits the representation

$$f(t) \ = \ f(+0) + \int_0^t \ f'(s) \ ds,$$

where f' denotes the derivative of f.

We shall repeatedly use the following simple result.

Proposition 2.7. *In a Hausdorff linear topological space a sequence $\{x_n\}$ converges to an element x if and only if each subsequence $\{y_n\}$ contains a further subsequence $\{z_n\}$ which converges to x.*

We conclude this section with several results concerning pointwise convergence that will be needed in the sequel. We include their proofs for lack of a convenient reference. We recall that a sequence $\{f_n\}_{n=1}^\infty$ of measurable functions converges locally in measure, that is, converges in measure on every set of finite measure, to the measurable function f if and only if every subsequence of the sequence $\{f_n\}$ contains a further subsequence which converges to f almost everywhere. We observe that for sequences of decreasing functions on $(0,\infty)$, local convergence in measure coincides with almost everywhere convergence.

Proposition 2.8. *Let $0 < \alpha \le \infty$. If $g,\ g_n,\ n = 1,2,\dots$ are real-valued, decreasing functions on $(0,\alpha)$, then $g_n \to g$ locally in measure on $(0,\alpha)$ if and only if $g_n \to g$ m-almost everywhere on $(0,\alpha)$.*

Proof. It is not difficult to see that the sequence $\{g_n\}_{n=1}^\infty$ is uniformly bounded on each interval of the form $[a,b]$, with $0 < a < b < \alpha$. The bounded convergence theorem of Lebesgue now implies that

$$\int_a^b g_n(t)dt \to \int_a^b g(t)dt$$

for all $0 < a < b < \alpha$. The argument of [16], Proposition 40 now shows that $g_n(s) \to g(s)$ at every point of continuity of the function g, and therefore almost everywhere, since g is decreasing. □

Proposition 2.9. *If $f_n, f \in Q_0$ and if $\lim_n f_n(t) = f(t)$, for all $t > 0$, then $f_n' \to f'$ m-almost everywhere on $(0, \infty)$.*

Proof. If $\phi \in Q_0$, then by concavity we have $\phi'(t) \leq \phi(t)/t$, for almost all $t > 0$. It follows that the sequence $\{f_n'\}$ of decreasing functions is uniformly bounded on every interval $[a, b]$, where $0 < a < b < \infty$. Let $\{g_n\}$ be a subsequence of $\{f_n\}$. By the Helly Selection Theorem, [25] VIII.4, Lemma 2, there exists a subsequence $\{h_n\}$ of $\{g_n\}$ and decreasing function k such that $h_n'(t) \to k(t)$ for all $t > 0$. Therefore

$$\int_a^b k(t)\ dt \ = \ \lim_n \int_a^b h_n'(t)\ dt \ = \ f(b) - f(a),$$

for all $0 < a < b$. Consequently, $k(t) = f'(t)$ m-almost everywhere on $(0, \infty)$ and by the remarks preceding Proposition 2.8, it follows that $f_n' \to f'$ locally in measure. The conclusion of the proposition now follows from Proposition 2.8. □

We recall [21] that the function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ is called *quasiconcave* if ψ is increasing and if the function $t \to \psi(t)/t,\ t > 0$, is decreasing. We denote by $\overline{Q}$ the cone of all quasiconcave functions on $\mathbb{R}^+$.

Proposition 2.10. *Let $0 < \alpha \leq \infty$. If $f_n,\ f \in \overline{Q},\ n = 1, 2, \ldots,$ and if $f_n(t) \to f(t)$ m-almost everywhere on $(0, \alpha)$, then $f_n(t) \to f(t)$ for all $t \in (0, \alpha)$.*

Proof. For definiteness, we consider only the case that $\alpha = \infty$, since the same argument applies to the case $\alpha < \infty$. Let $0 < \epsilon < 1$ be given. By assumption, there exists a number $a > 1/\epsilon$ such that $f_n(a) \to f(a)$ and so there exists a constant $c(\epsilon)$ such that

$$0 \leq f_n(a) \leq c(\epsilon), \quad n = 1, 2, \ldots\ .$$

It follows that

$$f_n(t)/t \leq f_n(\epsilon)/\epsilon \leq f_n(a)/\epsilon \leq c(\epsilon)/\epsilon,$$

for all $t \in [\epsilon, 1/\epsilon)$ and all $n = 1, 2, \ldots$. If

$$\delta < \epsilon^2/\max\{c(\epsilon), f(\epsilon)\},$$

then it follows that

$$|f_n(t) - f_n(s)| < \epsilon, \quad |f(t) - f(s)| < \epsilon,$$

for all $n = 1, 2, \ldots$ and all $t, s \in [\epsilon, 1/\epsilon)$ such that $|t - s| < \delta$. Let $\{I_k\}_{k \in J}$ be a partition of $[\epsilon, 1/\epsilon)$ into finitely many subintervals of length at most δ. By assumption, there exist $t_i \in I_i, i \in J$ such that $f_n(t_i) \to f(t_i)$ for each $i \in J$ and consequently, there exists $n = n(\epsilon)$ such that

$$\sup_{i \in J} |f_n(t_i) - f(t_i)| < \epsilon$$

for all $n \geq n(\epsilon)$. If $0 < t \in I_i$, then

$$|f_n(t) - f(t)| \leq |f_n(t) - f_n(t_i)| + |f_n(t_i) - f(t_i)| + |f(t_i) - f(t)| < 3\epsilon$$

for all $n \geq n(\epsilon)$, and this suffices to complete the proof of the Proposition. □

Corollary 2.11. *Let $0 < \alpha \leq \infty$. If $f_n,\ f \in \overline{Q},\ n = 1, 2, \ldots,$ then $f_n(t) \to f(t)$ locally in measure on $(0, \alpha)$ if and only if $f_n(t) \to f(t)$ for all $t \in (0, \alpha)$.*

The Corollary follows immediately from Proposition 2.8 and Proposition 2.10

3. Properties of a Banach Couple related to Local Uniform Convexity and Kadec-Klee Properties

In this section, we introduce and study properties of a Banach couple $\mathbf{E}$ which are key to showing local uniform convexity and Kadec-Klee type properties in its K-interpolation spaces. These properties are expressed in terms of the K-functional of the couple and play a central role in what follows.

Throughout this section, $\mathbf{E} = (E_0, E_1)$ will denote a Banach couple [5], [21], and set $\|\cdot\|_i = \|\cdot\|_{E_i},\ i = 0, 1$. Let $S(\mathbf{E})$ and $I(\mathbf{E})$ denote the sum $E_0 + E_1$ and the intersection $E_0 \cap E_1$, respectively, with the usual norms:

$$\begin{aligned} \|x\|_{S(\mathbf{E})} &= \inf\{\|x^0\|_0 + \|x^1\|_1 : x = x^0 + x^1,\ x^0 \in E_0,\ x^1 \in E_1\}, \\ \|x\|_{I(\mathbf{E})} &= \max\{\|x\|_0,\ \|x\|_1\}. \end{aligned}$$

The well-known K-functional $K(t, x; \mathbf{E})$ of J. Peetre is defined for $x \in S(\mathbf{E})$ and $t > 0$ by setting

$$\begin{aligned} K(t, x; \mathbf{E}) &= K(t, x; E_0, E_1) \\ &= \inf\{\|x^0\|_0 + t\|x^1\|_1 :\ x = x^0 + x^1,\ x^0 \in E_0,\ x^1 \in E_1\}. \end{aligned}$$

It will be convenient to write $K(t, x) = K(t, x; \mathbf{E})$, and this will cause no confusion in the sequel. For each $x \in S(\mathbf{E})$, the function $K(\cdot, x)$ is concave and for every fixed $t > 0$, the K-functional $K(t, \cdot)$ is a norm on

$S(\mathbf{E})$, equivalent to the given norm of $S(\mathbf{E})$. This follows immediately by observing that

$$\min(1,t)\|x\|_{S(\mathbf{E})} \;\leq\; K(t,x) \;\leq\; \max(1,t)\|x\|_{S(\mathbf{E})},$$

for all $t > 0$ and $x \in S(\mathbf{E})$.

We recall some basic properties of the K-functional which may be found in [4], [5] or can be easily proved. If E is a Banach space, if $H \subseteq E$ is a linear subspace of E and if $x \in E$, then we denote by $d_E(x,H)$ the distance between x to H.

Proposition 3.1. *If* $\mathbf{E} = (E_0, E_1)$ *is a Banach couple and if* $x, x_n \in S(\mathbf{E})$, $n = 1, 2, \ldots$, *then*

(a) $\dfrac{K(t,x;E_0,E_1)}{t} = K(1/t,x;E_1,E_0)$, *for all* $t > 0$.

(b) $\lim_{t\to 0} K(t,x;\mathbf{E}) = d_{S(\mathbf{E})}(x,E_1)$.

(c) $\lim_{t\to\infty} \dfrac{K(t,x;\mathbf{E})}{t} = d_{S(\mathbf{E})}(x,E_0)$.

(d) *If* $x_n \to x$ *in* $S(\mathbf{E})$ *then* $K(t,x_n) \to K(t,x)$, *for all* $t > 0$.

(e) *If* $x \in I(\mathbf{E})$ *then, for all* $t > 0$,

$$K(t,x;\mathbf{E}) \leq \min(1,t)\|x\|_{I(\mathbf{E})}.$$

We denote by $\overline{E}_i$ the closure of E_i in $S(\mathbf{E})$, $i = 0, 1$. An immediate consequence of the preceding Proposition is that $x \in \overline{E}_0$ (respectively, $x \in \overline{E}_1$) if and only if

$$\lim_{t\to\infty} t^{-1}K(t,x;\mathbf{E}) = 0, \quad \text{(respectively,} \quad \lim_{t\to 0} K(t,x;\mathbf{E}) = 0).$$

It is now convenient to introduce the $K-$functionals defined by setting

$$K_p(t,x;\mathbf{E}) = K_p(t,x) = \inf\{(\|x^0\|_0^p + t^p\|x^1\|_1^p)^{1/p} :\; x_i \in E_i,\; x = x^0 + x^1\},$$

for each $t > 0$, $1 \leq p \leq \infty$ and $x \in S(\mathbf{E})$. See, for example, [22], [17]. It is shown in [17] that if $1 \leq p \leq q \leq \infty$ and if $1/r = 1/p - 1/q$, then

$$\begin{aligned} K_p(t,x) &= \inf_{0<s<\infty} \left(1 + (t/s)^r\right)^{1/r} K_q(s,x), \\ K_q(t,x) &= \sup_{0<s<\infty} \left(1 + (s/t)^r\right)^{-1/r} K_p(s,x). \end{aligned}$$

For every $t > 0$, the norm $K_p(t,\cdot)$ is equivalent to the norm $K_1(t,\cdot) = K(t,\cdot)$ with constant not exceeding $2^{1-1/p}$. For every $x \in S(\mathbf{E})$, the function $K_p(\cdot,x)$ is quasiconcave, but is not in general a concave function if $1 < p \leq \infty$.

If $1 \le p \le \infty$, the couple $\mathbf{E} = (E_0, E_1)$ is said to be *p-exact* if, for every $x \in S(\mathbf{E})$, and for every $t > 0$, there exist $x^0 \in E_0$, $x^1 \in E_1$ such that $x = x^0 + x^1$ and

$$K_p(t,x) \; = \; (\|x^0\|_0^p + t^p \|x^1\|_1^p)^{1/p},$$

if $1 \le p < \infty$, and

$$K_p(t,x) \; = \; \max\{\|x^0\|_0, \|x^1\|_1\},$$

if $p = \infty$. In the case that $p = 1$, we shall say simply that the couple $\mathbf{E}$ is exact.

In what follows, we will always suppose that $\mathcal{T}$ is a Hausdorff linear topology on $S(\mathbf{E})$ that is weaker than the norm topology. To smoothen the presentation, it will be convenient to say that the Banach couple $\mathbf{E}$ is *$\mathcal{T}$-closed* if and only if the unit balls of E_0, E_1 are sequentially $\mathcal{T}$-closed in $S(\mathbf{E})$ and at least one is sequentially $\mathcal{T}$-compact. This condition is necessarily satisfied in a wide class of Banach couples of particular interest. For example, it is not difficult to verify that if E_0, E_1 are symmetric Banach function spaces on $\mathbb{R}^+$ of which one has the Fatou property (or is maximal, in the terminology of [21]) and the other is the dual of a separable space, and if $\mathcal{T}$ is the weak topology of pointwise convergence on $L_1 \cap L_\infty$, then the couple $\mathbf{E}$ is $\mathcal{T}$-closed. In particular, these conditions are satisfied by the familiar Lebesgue couple (L_1, L_∞).

Proposition 3.2. *If the Banach couple $\mathbf{E} = (E_0, E_1)$ is $\mathcal{T}$-closed, then $\mathbf{E}$ is p-exact for every $1 \le p \le \infty$.*

Proof. Without loss of generality we can assume that the unit ball of E_0 is sequentially $\mathcal{T}$-compact. Let $x \in S(\mathbf{E})$ and fix $t > 0$ and $1 \le p \le \infty$. Select $x_n^i \in E_i$, $i = 0, 1$, such that $x = x_n^0 + x_n^1$ and

$$K_p(t,x) = (\|x_n^0\|_0^p + t^p \|x_n^1\|_1^p)^{1/p} + o(1). \tag{3.1}$$

From (3.1), it follows that the sequences $\{x_n^0\}, \{x_n^1\}$ are bounded in E_0, E_1 respectively. By passing to a subsequence if necessary, and using the fact that the unit ball in E_0 is sequentially $\mathcal{T}$-compact, we may assume that there exists $x^0 \in E_0$ such that $x_n^0 \to_{\mathcal{T}} x^0$ and that

$$\|x^0\|_0 \le \underline{\lim}_n \|x_n^0\|_0.$$

Since the unit ball in E_1 is sequentially $\mathcal{T}$-closed in $S(\mathbf{E})$, it follows that $x^1 := x - x^0 \in E_1$ and

$$\|x^1\|_1 \le \underline{\lim}_n \|x_n^1\|_1.$$

It follows that

$$\begin{aligned}(\|x^0\|_0^p + t^p\|x^1\|_1^p)^{1/p} &\le (\underline{\lim}_n\|x_n^0\|_0^p + t^p\underline{\lim}_n\|x_n^1\|_1^p)^{1/p}\\ &\le \underline{\lim}_n\left(\|x_n^0\|_0^p + t^p\|x_n^1\|_1^p)\right)^{1/p}\\ &= \lim_n\left(\|x_n^0\|_0^p + t^p\|x_n^1\|_1^p\right)^{1/p} = K_p(t,x),\end{aligned}$$

and this suffices to complete the proof of the Proposition. □

Definition 3.3. *Let E be a linear subspace of $S(\mathbf{E})$ and let $\emptyset \neq T \subseteq (0,\infty)$. The Banach couple $\mathbf{E}$ is said to have property $(C_{\mathcal{T}})$ on the linear subspace E, written $\mathbf{E} \in (C_{\mathcal{T}}|E)$ relative to T, if and only if whenever $x_n, x \in E$ satisfy $x_n \to_{\mathcal{T}} x$ then it follows that*

$$K(t,x) \le \underline{\lim}_n K(t,x_n)$$

for all $t \in T$.

We remark that in the preceding definition, it is not assumed that the linear subspace E be closed.

Proposition 3.4. *If the Banach couple $\mathbf{E} = (E_0, E_1)$ is $\mathcal{T}$-closed, then $\mathbf{E} \in (C_{\mathcal{T}}|S(\mathbf{E}))$ relative to $(0,\infty)$.*

Proof. Without loss of generality we can assume that the unit ball of E_0 is sequentially $\mathcal{T}$-compact. Let $x_n, x \in S(\mathbf{E}), n = 1,2,\ldots,$ and suppose that $x_n \to_{\mathcal{T}} x$. Fix a number $t > 0$. If $\underline{\lim}_n K(t,x_n) = \infty$, there is nothing to prove. We may assume then that there exists a subsequence $\{y_n\} \subseteq \{x_n\}$ such that

$$\lim_n K(t,y_n) = \underline{\lim}_n K(t,x_n) < \infty.$$

By Proposition 2.2, it follows that $\mathbf{E}$ is exact and so for each $n = 1,2,\ldots,$ there exists $y_n^i \in E_i, i = 0,1$ such that $y_n = y_n^0 + y_n^1$ and

$$K(t,y_n) = \|y_n^0\|_0 + t\|y_n^1\|_1. \tag{2.2}$$

Since the sequences $\{y_n^0\}, \{y_n^1\}$ are bounded in E_0, E_1 respectively, and by passing to subsequences if necessary, it follows by an argument similar to that in the proof of Proposition 3.2 that there exist $y^i \in E_i, i = 1,2$ such that $x = y^0 + y^1$ and

$$y_n^i \to_{\mathcal{T}} y^i \quad \text{and} \quad \|y^i\|_i \le \underline{\lim}_n\|y_n^i\|_i, \quad i = 1,2.$$

It follows that

$$\begin{aligned}K(t,x) &\le \|y^0\|_0 + t\|y^1\|_1 \le \underline{\lim}_n\|y_n^0\|_0 + t\underline{\lim}_n\|y_n^1\|_1\\ &\le \underline{\lim}_n(\|y_n^0\|_0 + t\|y_n^1\|_1) \le \lim_n K(t,y_n) = \underline{\lim}_n K(t,x_n),\end{aligned}$$

and this suffices to complete the proof. □

Definition 3.5. *Let E be a linear subspace of $S(\mathbf{E})$ and let $\emptyset \neq T \subseteq (0,\infty)$. The Banach couple $\mathbf{E}$ is said to have property $D_{\mathcal{T}}$ on the subspace E, written $\mathbf{E} \in (D_{\mathcal{T}}|E)$, relative to T if and only if whenever $x_n, x \in E$ satisfy $x_n \to_{\mathcal{T}} x$ and $K(t,x_n) \to K(t,x)$ for all $t \in T$ then it follows that*

$$x_n \longrightarrow_{S(\mathbf{E})} x.$$

Theorem 3.6. *Let $\mathbf{E} = (E_0, E_1)$ be a Banach couple and assume that $\mathbf{E}$ is $\mathcal{T}$-closed.*

(i) *If E_0, E_1 have the Kadec-Klee property with respect to $\mathcal{T}$ then $\mathbf{E} \in (D_{\mathcal{T}}|S(\mathbf{E}))$ relative to any non-empty subset of $(0,\infty)$.*
(ii) *If E_0 has the Kadec-Klee property with respect to $\mathcal{T}$ then $\mathbf{E} \in (D_{\mathcal{T}}|\overline{E_0})$, relative to any non-empty subset of $(0,\infty)$ having ∞ as a limit point.*
(iii) *If E_1 has the Kadec-Klee property with respect to $\mathcal{T}$ then $\mathbf{E} \in (D_{\mathcal{T}}|\overline{E_1})$, relative to any non-empty subset of $(0,\infty)$ having 0 as a limit point.*

Proof. We make first some preliminary remarks. Suppose that $x_n, x \in S(\mathbf{E})$ satisfy $x_n \to_{\mathcal{T}} x$ and $K(t_0, x_n) \to K(t_0, x)$ for some $t_0 > 0$. By Proposition 3.2 the couple $\mathbf{E}$ is exact, and so for each $n = 1, 2, \ldots,$, there exist $x_n^i \in E_i, i = 0, 1$ such that $x_n = x_n^0 + x_n^1$ and

$$K(t_0, x_n) = \|x_n^0\|_0 + t_0 \|x_n^1\|_1. \tag{3.3}$$

From this, it follows that

$$\overline{\lim}_n \|x_n^0\|_0 \le K(t_0, x), \quad \overline{\lim}_n \|x_n^1\|_1 \le \frac{K(t_0,x)}{t_0}.$$

By passing to subsequences if necessary, we may, as in the proof of Proposition 3.2, assume that there exist $x^i \in E_i$, $i = 0, 1$ such that $x = x^0 + x^1$ and $x_n^i \to_{\mathcal{T}} x^i$, $i = 0, 1$, and such that

$$\|x^i\|_i \le \underline{\lim}_n \|x_n^i\|_i, \quad i = 0, 1. \tag{3.4}$$

Consequently

$$\overline{\lim}_n \|x_n^0\|_0 = \overline{\lim}_n \left(K(t_0, x_n) - t_0 \|x_n^1\|_1\right) \le K(t_0, x) - t_0 \|x^1\|_1 \le \|x^0\|_0,$$

and

$$\overline{\lim}_n \|x_n^1\|_1 = t_0^{-1} \overline{\lim}_n \left(K(t_0, x_n) - \|x_n^0\|_0\right) \le t_0^{-1} \left(K(t_0, x) - \|x^0\|_0\right) \le \|x^1\|_1.$$

From these inequalities and (2.4), it follows that $\|x_n^i\|_i \to \|x^i\|_i$, for $i = 1, 2$.

If now we assume that E_0, E_1 have the Kadec-Klee property relative to $\mathcal{T}$, then it follows from the preceding paragraph that $\|x_n^i - x_i\|_i \to 0, i = 1, 2$. This implies that $x_n \to x$ holds in $S(\mathbf{E})$. Via Proposition 1.7, this shows that $\mathbf{E} \in (D_{\mathcal{T}}|S(\mathbf{E}))$ relative to any subset of $(0,\infty)$ containing the point t_0 and this suffices to complete the proof of (i).

To establish the assertion of (ii), we assume further that $x_n, x \in \overline{E_0}$. If E_0 has the Kadec-Klee property relative to $\mathcal{T}$, then we obtain that $\|x_n^0 - x^0\|_0 \to 0$. Using (3.3) and (3.4), observe now that

$$\begin{aligned} \overline{\lim}_n \|x - x_n\|_{S(\mathbf{E})} &\leq \overline{\lim}_n \left(\|x^0 - x_n^0\|_0 + \|x^1 - x_n^1\|_1\right) \\ &\leq \|x^1\|_1 + \overline{\lim}_n \|x_n^1\|_1 \leq 2K(t_0, x)/t_0. \end{aligned}$$

Consequently, if T is any non-empty subset of $(0,\infty)$ having ∞ as a limit point and if $t_0 \in T$ is arbitrary, then it follows from the fact that $x \in \overline{E_0}$ and Proposition 3.1(c) that $K(t_0, x)/t_0 \to 0$ as $t_0 \to \infty$. A simple subsequence argument combined with Proposition 2.7 now suffices to complete the proof of (ii).

Finally, the proof of (iii) is almost identical to that of (ii), using Proposition 3.1(b). □

Definition 3.7. *Let E be a linear subspace of $S(\mathbf{E})$ and let $\emptyset \neq T \subseteq (0,\infty)$. If $1 \leq p < \infty$, then the Banach couple $\mathbf{E}$ is said to have property (DGL^p) on the subspace E, written $\mathbf{E} \in (DGL^p|E)$, relative to T if and only if whenever $x, x_n \in E, n = 1, 2, \ldots,$ satisfy*

$$K_p(t, x_n) \longrightarrow_n K_p(t, x), \quad K_p(t, x + x_n) \longrightarrow_n 2K_p(t, x)$$

for all $t \in T$, it follows that $x_n \to x$ in $S(\mathbf{E})$.

In the case that $p = 1$, it will be convenient to refer to property (DGL) rather than (DGL^1).

If (Ω, P) is a probability space, then it was shown implicitly by Davis, Ghoussoub and Lindenstrauss ([14], Proposition 1.1) that the Banach couple $(L_1(\Omega, P), L_\infty(\Omega, P))$ has the property $(DGL|L_1(\Omega, P))$ relative to $(0,1)$, and this is the source of our terminology.

Theorem 3.8. *Assume that the Banach couple $\mathbf{E} = (E_0, E_1)$ is exact.*

(i) *If E_0, E_1 are locally uniformly convex, then $\mathbf{E} \in (DGL|S(\mathbf{E}))$ relative to $(0,\infty)$.*
(ii) *If E_0 is locally uniformly convex, then $\mathbf{E} \in (DGL|\overline{E_0})$ relative to any set of the form (a,∞) for some $a \geq 0$.*
(iii) *If E_1 is locally uniformly convex, then $\mathbf{E} \in (DGL|\overline{E_1})$ relative to any set of the form $(0,a)$ for some $a > 0$.*

It will be convenient to isolate several of the steps of the proof as follows.

Lemma 3.9. *Assume that the couple* $\mathbf{E} = (E_0, E_1)$ *is exact. Let* $0 \neq x \in \overline{E_0}$. *For all* $\epsilon > 0$ *and* $a > 0$, *there exists* $s_0 > a$ *such that* $K(\cdot, x)$ *is differentiable at* s_0 *and if*

$$K(s_0, x) = \|x^0\|_0 + s_0 \|x^1\|_1,$$

where $x^i \in E_i,\ i = 0, 1,$ *and* $x = x^0 + x^1$, *then*

$$\|x^0\|_0 > 0,\ \|x^1\|_1 < \frac{\epsilon}{3}.$$

Proof. Let $0 \neq x \in \overline{E_0}$, $\epsilon > 0$ and $a > 0$. If $t_0 > 0$ is given, we may assume that $\epsilon < \frac{K(t_0,x)}{t_0}$. Since $\lim_{t\to\infty} \frac{K(t,x)}{t} = d_{S(\mathbf{E})}(x, E_0) = 0$, there exists $s_0 > a$ such that $K(\cdot, x)$ is differentiable at s_0 and

$$\frac{K(s_0, x)}{s_0} < \frac{\epsilon}{3}.$$

Now if

$$K(s_0, x) = \|x^0\|_0 + s_0 \|x^1\|_1,$$

where $x^i \in E_i,\ i = 0, 1,$ and $x = x^0 + x^1$, then

$$\|x^1\|_1 = \frac{K(s_0, x)}{s_0} - \frac{\|x^0\|_0}{s_0} \leq \frac{K(s_0, x)}{s_0} < \frac{\epsilon}{3},$$

and, from $K(t_0, x) \leq \|x^0\|_0 + t_0 \|x^1\|_1$, we have

$$\|x^0\|_0 \geq t_0 \left(\frac{K(t_0, x)}{t_0} - \|x^1\|_1 \right) > t_0 \left(\epsilon - \frac{\epsilon}{3} \right) > 0. \qquad \square$$

Lemma 3.10 *Assume that the Banach couple* $\mathbf{E} = (E_0, E_1)$ *is exact. Let* (a, b) *be a non-empty open interval of* $\mathbb{R}^+$ *and let* $x, x_n \in S(\mathbf{E})$, $n = 1, 2, \ldots$. *If* $K(t, x_n) \to_n K(t, x)$ *for all* $t \in (a, b)$, *if* $K(\cdot, x)$ *is differentiable at* $s_0 \in (a, b)$ *and if*

$$K(s_0, x) = \|x^0\|_0 + s_0 \|x^1\|_1, \quad K(s_0, x_n) = \|x_n^0\|_0 + s_0 \|x_n^1\|_1, \text{ for } n = 1, 2, \ldots$$

where $x^i, x_n^i \in E_i, i = 0, 1,$ *and* $x = x^0 + x^1$, $x_n = x_n^0 + x_n^1$, *then*

$$\|x_n^0\|_0 \longrightarrow \|x^0\|_0 \qquad \text{and} \qquad \|x_n^1\|_1 \longrightarrow \|x^1\|_1.$$

Proof. Since $\{x_n^0\}$, $\{x_n^1\}$ is bounded in E_0, E_1, respectively, there exists a subsequence $\{y_n\}$ of $\{x_n\}$ such that

$$y_n = y_n^0 + y_n^1,$$

where $\{y_n^0\}$, $\{y_n^1\}$ is the corresponding subsequence of $\{x_n^0\}$, $\{x_n^1\}$, respectively, and

$$\|y_n^0\|_0 \longrightarrow A \qquad \text{and} \qquad \|y_n^1\|_1 \longrightarrow B,$$

for some $A, B \geq 0$. From the assumption $K(t, x_n) \to_n K(t, x)$ for all $t \in (a, b)$, we have

$$K(s_0, x) = \|x^0\|_0 + s_0\|x^1\|_1 = \lim_n K(s_0, y_n) = \lim_n(\|y_n^0\|_0 + s_0\|y_n^1\|_1) = A + s_0 B$$

and

$$K(t, x) = \lim_n K(t, y_n) \leq \lim_n(\|y_n^0\|_0 + t\|y_n^1\|_1) = A + tB,$$

for all $t \in (a, b)$. Since

$$K(s_0, x) = \|x^0\|_0 + s_0\|x^1\|_1 \quad \text{and} \quad K(t, x) \leq \|x^0\|_0 + t\|x^1\|_1$$

for all $t \in (a, b)$, and since $K(\cdot, x)$ is differentiable at s_0, it follows that $A = \|x^0\|_0$ and $B = \|x^1\|_1$. Via Proposition 2.7, this suffices to complete the proof of the lemma. □

Proof of Theorem 3.8. Proof of (ii). Let T be a a non-empty set of the form (a, ∞) for some $a \geq 0$, let $0 \neq x \in \overline{E_0}$ and let $x_n \in S(\mathbf{E}), n = 1, 2, \ldots,$ be such that

$$K(t, x_n) \longrightarrow K(t, x) \quad \text{and} \quad K(t, x_n + x) \longrightarrow 2K(t, x), \qquad \text{for all } t \in T.$$

By Lemma 3.9, for $\epsilon > 0$, there exists $s_0 \in T$ such that $K(\cdot, x)$ is differentiable at s_0,

$$K(s_0, x) = \|x^0\|_0 + s_0\|x^1\|_1,$$

where $x^0 \in E_0,\ x^1 \in E_1,\ x = x^0 + x^1$, and

$$\|x^0\|_0 > 0,\ \|x^1\|_1 < \frac{\epsilon}{3}.$$

By Lemma 3.10, if

$$K(s_0, x_n) = \|x_n^0\|_0 + s_0\|x_n^1\|_1,$$

where $x_n^0 \in E_0,\ x_n^1 \in E_1, x_n = x_n^0 + x_n^1, n = 1, 2, \ldots,$ then

$$\|x_n^0\|_0 \longrightarrow \|x^0\|_0 \qquad \text{and} \qquad \|x_n^1\|_1 \longrightarrow \|x^1\|_1.$$

It follows that

$$x_n^0 \neq 0 \quad \text{and} \quad \min(\|x^0\|_0, \|x_n^0\|_0) > \|x^0\|_0/2$$

for all n sufficiently large. Applying the triangle inequality to $\|\cdot\|_1$, we obtain

$$\begin{aligned} K(s_0, x + x_n) &\leq \|x^0 + x_n^0\|_0 + s_0\|x^1 + x_n^1\|_1 \\ &\leq \|x^0\|_0 + \|x_n^0\|_0 - \delta_n + s_0\|x^1\|_1 + s_0\|x_n^1\|_1 \\ &= K(s_0, x) + K(s_0, x_n) - \delta_n \end{aligned}$$

where, setting $\overline{z} = z/\|z\|$ if $z \neq 0$,

$$\delta_n = \delta_{\overline{x^0}}\left(\left\|\frac{\overline{x^0} - \overline{x_n^0}}{2}\right\|_0\right) \cdot \min(\|x^0\|_0, \|x_n^0\|_0), \quad n = 1, 2, \dots .$$

Since $K(s_0, x_n) \to K(s_0, x)$ and $K(s_0, x_n + x) \to 2K(s_0, x)$, it follows that $\delta_n \to 0$ and so, using the assumption that E_0 is locally uniformly convex, we have that $\|\overline{x^0} - \overline{x_n^0}\|_0 \to 0$. Now

$$\begin{aligned} \|x^0 - x_n^0\|_0 &= \left\| \, \|x^0\|_0 \left(\overline{x^0} - \overline{x_n^0}\right) + \left(\|x^0\|_0 - \|x_n^0\|_0\right) \overline{x_n^0} \, \right\|_0 \\ &\leq \|x^0\|_0 \, \|\overline{x^0} - \overline{x_n^0}\|_0 + \left|\|x^0\|_0 - \|x_n^0\|_0\right| \, \|\overline{x_n^0}\|_0 \longrightarrow 0, \end{aligned}$$

and thus

$$\begin{aligned} \|x - x_n\|_{S(\mathbf{E})} &\leq \|x^0 - x_n^0\|_0 + \|x^1 - x_n^1\|_1 \\ &\leq \|x^0 - x_n^0\|_0 + \|x^1\|_1 + \|x_n^1\|_1 \leq \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon, \end{aligned}$$

for all sufficiently large n, and this suffices to complete the proof of (i).

Proof of (iii). Since

$$K(1/t, x; E_1, E_0) = \frac{K(t, x; E_0, E_1)}{t}$$

for all $t > 0$, we obtain (iii) by interchanging E_0 with E_1 in (ii).

Proof of (i). Suppose that $x, x_n \in S(\mathbf{E})$, $n = 1, 2, \dots$, satisfy

$$K(t, x_n) \to K(t, x), \quad K(t, x + x_n) \to 2K(t, x)$$

for all $t \in (0, \infty)$. By the proofs of (ii) and (iii) preceding, we may assume that $x \notin \overline{E_0}$ and $x \notin \overline{E_1}$. Let $s_0 \in (0, \infty)$ be such that $K(\cdot, x)$

is differentiable at s_0 and let

$$K(s_0, x) = \|x^0\|_0 + s_0\|x^1\|_1, \quad K(s_0, x_n) = \|x_n^0\|_0 + s_0\|x_n^1\|_1,$$

where $x^0 \in E_0, x^1 \in E_1, x = x^0 + x^1$ and $x_n^0 \in E_0, x_n^1 \in E_1, x_n = x_n^0 + x_n^1, n = 1, 2, \ldots$. It follows that $\|x^0\|_0 > 0$ and $\|x^1\|_1 > 0$. Moreover, by Lemma 3.10,

$$\|x_n^0\|_0 \longrightarrow \|x^0\|_0, \qquad \|x_n^1\|_1 \longrightarrow \|x^1\|_1,$$

and hence

$$\min(\|x^0\|_0, \|x_n^0\|_0) > \|x^0\|_0/2, \qquad \min(\|x^1\|_1, \|x_n^1\|_1) > \|x^1\|_1/2,$$

for all n sufficiently large. We obtain that

$$\begin{aligned} K(s_0, x + x_n) &\leq \|x^0 + x_n^0\|_0 + s_0\|x^1 + x_n^1\|_1 \\ &\leq \|x^0\|_0 + \|x_n^0\| - \delta_n^0 + s_0\|x^1\|_1 + s_0\|x_n^1\|_1 - s_0\delta_n^1 \\ &= K(s_0, x) + K(s_0, x_n) - \delta_n^0 - s_0\delta_n^1 \end{aligned}$$

where

$$\delta_n^i = \delta_{\overline{x^i}}\left(\left\|\frac{\overline{x^i} - \overline{x_n^i}}{2}\right\|_i\right) \cdot \min(\|x^i\|_i, \|x_n^i\|_i), \quad i = 0, 1.$$

Since $K(s_0, x_n) \to K(s_0, x)$ and $K(s_0, x_n + x) \to 2K(s_0, x)$, it follows that $\delta_n^i \to 0, i = 0, 1$. The assumption that E_0, E_1 are locally uniformly convex now implies that $\|\overline{x^i} - \overline{x_n^i}\|_i \to 0, i = 1, 2$. Therefore

$$\|x - x_n\|_{S(\mathbf{E})} \leq \|x^0 - x_n^0\|_0 + \|x^1 - x_n^1\|_1 \longrightarrow 0.$$

This suffices to complete the proof of the Theorem. $\square$

The preceding theorem shows that if either or both members of an exact Banach couple are locally uniformly convex, then the couple itself automatically possesses a certain (DGL)-property. It is perhaps important to note that there are exact couples of particular interest which have the (DGL)-property even though neither element of the couple is locally uniformly convex. For example, neither of the spaces $E_0 = L_1[0, 1)$, $E_1 = L_\infty[0, 1)$ are locally uniformly convex, although, as noted above the couple $\mathbf{E} = (E_0, E_1)$ is exact and has the (DGL)-property on E_0 relative to $(0, 1)$. A wide class of such couples will be considered in a separate paper [12].

The (DGL)-property of a given couple $\mathbf{E}$ may be considered as key to exhibiting the existence of equivalent locally uniformly convex norms. The Proposition which follows expresses the core idea in the approach of [14], in the case of the Lebesgue couple $\mathbf{L}[0, 1)$.

Proposition 3.11. *If $L \subseteq S(\mathbf{E})$ is a linear subspace, if $\emptyset \neq T \subseteq (0,\infty)$ and if $\mathbf{E} \in (DGL|L)$ relative to T, then L admits a locally uniformly convex norm equivalent to that induced by $S(\mathbf{E})$.*

Proof. We set
$$|||x||| = \|K(\cdot,x)\|_2, \quad x \in S(\mathbb{E}),$$
where
$$\|f\|_2 := \left(\int_0^\infty \left[f(t)\min\left(1,t^{-2}\right)\right]^2 dt\right)^{1/2}, \quad f \in Q.$$
That $|||\cdot|||$ is equivalent to the norm $K(1,\cdot)$ on $S(\mathbf{E})$ follows immediately from the inequalities
$$\min(1,t)K(1,x) \leq K(t,x) \leq \max(1,t)K(1,x), \quad t>0$$
for every $x \in S(\mathbf{E})$. To show that $|||\cdot|||$ is a locally uniformly convex norm on L, suppose that $x_n, x \in L, n = 1,2,\ldots$ satisfy
$$|||x_n||| \to |||x|||, \quad |||x+x_n||| \to 2|||x|||.$$
It will suffice to show that some subsequence of the sequence $\{x_n\}_{n=1}^\infty$ converges to x in the norm of $S(\mathbf{E})$. From the inequalities
$$\|K(\cdot,x_n+x)\|_2 \leq \|K(\cdot,x_n)+K(\cdot,x)\|_2 \leq \|K(\cdot,x_n)\|_2+\|K(\cdot,x)\|_2, \quad n=1,2,\ldots,$$
and noting that the left and right terms tend to $2\|K(\cdot,x)\|_2$, it follows that
$$\|K(\cdot,x_n)+K(\cdot,x)\|_2 \to 2\|K(\cdot,x)\|_2.$$
Since $\|\cdot\|_2$ is a locally uniformly convex norm, and therefore has the uniform strong majorant property (see Preliminaries), it follows that
$$\|K(\cdot,x_n)-K(\cdot,x)\|_2 \to 0,$$
and
$$\|K(\cdot,x_n+x)-K(\cdot,x_n)-K(\cdot,x)\|_2 \to 0.$$
Consequently,
$$\|K(\cdot,x_n+x)-2K(\cdot,x)\|_2 \to 0.$$
Passing to subsequences, we may assume that
$$K(\cdot,x_n) \to K(\cdot,x), \quad K(\cdot,x_n+x) \to 2K(\cdot,x)$$
locally in measure, and passing to further subsequences and applying Corollary 2.11, we may further assume that
$$K(t,x_n) \to K(t,x), \quad K(t,x_n+x) \to 2K(t,x)$$

holds for each $t > 0$. The proof of the Proposition now follows immediately from the fact that $\mathbf{E} \in (DGL|L)$ relative to T. □

That the assumptions of the preceding Proposition are satisfied in the special case of any couple of the form $\mathbf{L}(\Omega, \mathcal{P}) = (L_1(\Omega, \mathcal{P}), L_\infty(\Omega, \mathcal{P}))$ with $(\Omega, \mathcal{P})$ a probability space and $L = L_1(\Omega, \mathcal{P})$ is shown in [14], Proposition 2.1. Consequently, $L_1(\Omega, \mathcal{P})$ has an equivalent locally uniformly convex norm and this is precisely [14] Theorem 2.1. On the other hand, we will now show how the classical Kadec renorming theorem follows simply and directly from our approach.

Corollary 3.12 (*Kadec renorming theorem*). *Each separable Banach space admits an equivalent locally uniformly convex norm.*

Proof. Since every separable Banach space is isometrically embedded in the space $C[0,1]$ of continuous functions on $[0,1]$, it will suffice to show that $C = C[0,1]$ admits an equivalent locally uniformly convex norm. We denote by $AC[0,1]$ the space of absolutely continuous functions on the interval $[0,1]$, and define the space W by setting

$$W = \{w \in AC[0,1] : w' \in L_2[0,1]\}.$$

With norm defined by setting

$$\|w\|_W := \left(\|w'\|^2_{L_2[0,1]} + |w(0)|^2\right)^{1/2}, \quad w \in W,$$

the space $(W, \|\cdot\|_W)$ is easily seen to be a Hilbert space. It is simple to check that

$$\|w\|_\infty = \sup_{t \in [0,1]} |w(t)| \leq \sqrt{2}\|w\|_W$$

for all $w \in W$, so that W is continuously embedded in C. We let $\mathbf{E}$ be the Banach couple (C, W) and let $\mathcal{T}$ be the linear Hausdorff topology $\sigma(C, L_1[0,1])$ on $S(\mathbf{E})$ of pointwise convergence on $L_1[0,1]$. Since $S(\mathbf{E}) = C$, the unit ball of C is $\mathcal{T}$-closed in $S(\mathbf{E})$. Since W is a Hilbert space, its unit ball is (sequentially) $\sigma(W, W^*)$-compact, and since the restriction of $\mathcal{T}$ to W is weaker than the weak topology $\sigma(W, W^*)$, it follows that the unit ball of W is sequentially $\mathcal{T}$-compact. Consequently the Banach couple $\mathbf{E} = (C, W)$ is $\mathcal{T}$-closed and so it follows from Proposition 3.2 that $\mathbf{E}$ is exact. By Theorem 3.8 (iii), it follows that $\mathbf{E} \in (DGL|\overline{W})$, and hence $\mathbf{E} \in (DGL|C)$, relative to any interval of the form $(0, a)$, for some $a > 0$. We use here the fact that W is $\|\cdot\|_\infty$-dense in C. It now follows from Proposition 3.11 that C admits an equivalent locally uniformly convex norm and this suffices to complete the proof. □

Theorem 3.13. *Assume that the Banach couple* $\mathbf{E} = (E_0, E_1)$ *is* p*-exact for some* $1 < p < \infty$.

(i) *If* E_0, E_1 *are locally uniformly convex, then* $\mathbf{E} \in (DGL^p|S(\mathbf{E}))$ *relative to any nonempty subset* T *of* $(0, \infty)$.

(ii) *If* E_0 *is locally uniformly convex, then* $\mathbf{E} \in (DGL^p|\overline{E_0})$ *relative to any nonempty subset* T *of* $(0, \infty)$ *having* ∞ *as a limit point.*

(iii) *If* E_1 *is locally uniformly convex, then* $\mathbf{E} \in (DGL^p|\overline{E_1})$ *relative to any nonempty subset* T *of* $(0, \infty)$ *having* 0 *as a limit point.*

Proof. Let $x_n, x, \in S(\mathbf{E}), n = 1, 2, \dots$ satisfy $K_p(t, x_n) \to K_p(t, x)$ and $K_p(t, x + x_n) \to 2K_p(t, x)$ for all $t \in T$. Since the couple $\mathbb{E}$ is p-exact, then, for any given $t \in T$ and $n = 1, 2, \dots$, there exist $x^0, x_n^0 \in E_0,\ x^1, x_n^1 \in E_1$ for which $x = x^0 + x^1,\ x_n = x_n^0 + x_n^1$, and

$$K_p(t, x) = (\|x^0\|_0^p + t^p\|x^1\|_1^p)^{1/p},\ K_p(t, x_n) = (\|x_n^0\|_0^p + t^p\|x_n^1\|_1^p)^{1/p}.$$

We define $\mathbf{u}, \mathbf{u_n}, \mathbf{w_n} \in l_p^{(2)},\ n = 1, 2, 3, \dots$, by setting

$$\mathbf{u} := (\|x^0\|_0, t\|x^1\|_1),\ \mathbf{u_n} := (\|x_n^0\|_0, t\|x_n^1\|_1),\ \mathbf{w_n} := (\|x^0 + x_n^0\|_0, t\|x^1 + x_n^1\|_1).$$

We obtain

$$K_p(t, x + x_n) \le \|\mathbf{w_n}\|_{l_p^{(2)}} \le \|\mathbf{u} + \mathbf{u_n}\|_{l_p^{(2)}} \le \|\mathbf{u}\|_{l_p^{(2)}} + \|\mathbf{u_n}\|_{l_p^{(2)}} \le K_p(t, x) + K_p(t, x_n)$$

for all $n = 1, 2, \dots$. Since the terms on the left and right hand sides of these inequalities have the same limits, we obtain that

$$\lim_{n\to\infty} \|\mathbf{w_n}\|_{l_p^{(2)}} = \lim_{n\to\infty} \|\mathbf{u} + \mathbf{u_n}\|_{l_p^{(2)}} = \|\mathbf{u}\|_{l_p^{(2)}} + \lim_{n\to\infty} \|\mathbf{u_n}\|_{l_p^{(2)}}.$$

Since $K_p(t, x_n) \to K_p(t, x)$, it follows that $\lim_{n\to\infty} \|\mathbf{u_n}\|_{l_p^{(2)}} \to \|\mathbf{u}\|_{l_p^{(2)}}$ and consequently it follows from the equality of the second two limits that $\|\mathbf{u} + \mathbf{u_n}\|_{l_p^{(2)}} \to 2\|\mathbf{u}\|_{l_p^{(2)}}$. Since the space $l_p^{(2)}$ is uniformly convex, it now follows that

$$\|\mathbf{u_n} - \mathbf{u}\|_{l_p^{(2)}} \to 0,$$

and this implies that

$$\|x_n^0\|_0 \to \|x^0\|_0 \quad \text{and} \quad \|x_n^1\|_1 \to \|x^1\|_1.$$

Since $\mathbf{0} \le \mathbf{w_n} \le \mathbf{u}_n + \mathbf{u},\ n = 1, 2, \dots$, it follows from the equality of the first two limits together with the fact that the space $l_p^{(2)}$ has the uniform strong majorant property that

$$\|\mathbf{u_n} + \mathbf{u} - \mathbf{w_n}\|_{l_p^{(2)}} \to 0.$$

This now implies that

$$\|x^0 + x_n^0\|_0 \to 2\|x^0\|_0 \quad \text{and} \quad \|x^1 + x_n^1\|_1 \to 2\|x^1\|_1.$$

Proof of (i). Using the assumption that E_0, E_1 are locally uniformly convex, it follows that $x_n^0 \to x^0$ in E_0 and $x_n^1 \to x^1$ in E_1. Consequently

$$x_n = x_n^0 + x_n^1 \to x^0 + x^1 = x$$

in $E_0 + E_1$ and this completes the proof of (i).

Proof of (ii). Suppose that $x_n, x \in \overline{E}_0$. The assumption that E_0 are locally uniformly convex now implies that $x_n^0 \to x^0$ in E_0. It follows that

$$\begin{aligned} \overline{\lim}_n \|x - x_n\|_{S(\mathbf{E})} &\leq \overline{\lim}_n \left(\|x^0 - x_n^0\|_0 + \|x^1 - x_n^1\|_1\right) \\ &\leq 2\|x^1\|_1 \leq 2K(t,x)/t. \end{aligned}$$

The assertion of (ii) now follows from the fact that ∞ is a limit point of T and using Proposition 3.1(c).

The proof of (iii) is almost identical to that of (ii), via Proposition 3.1(b). □

Corollary 3.14. *If E_0, E_1 are locally uniformly convex and if the Banach couple $\mathbf{E} = (E_0, E_1)$ is p-exact for some $1 < p < \infty$, then the space $S(\mathbf{E})$ equipped with the norm $K_p(1, \cdot)$ is locally uniformly convex.*

4. Interpolation of Local Uniform Convexity and Kadec-Klee Properties

In this section, we introduce a general framework to study interpolation of local uniform convexity and Kadec-Klee properties. Our principal results (Theorems 4.1, 4.2 below) establish general criteria in terms of a properties of a given Banach couple $\mathbf{E}$ and a K-interpolation functional Φ which guarantee that an interpolation space $\mathbf{E}_\Phi$ for the couple $\mathbf{E}$ derived from Φ has the Kadec-Klee property for some linear Hausdorff topology $\mathcal{T}$ on the space $S(\mathbf{E})$ (Theorem 4.1), or is locally uniformly convex (Theorems 4.2, 4.3). In subsequent sections, we shall show how these theorems may be refined for explicit K-interpolation functionals and Banach couples. We begin by introducing the notion of a K-interpolation functional.

Let $\mathcal{Q}$ be a cone of non-negative, increasing functions on $\mathbb{R}^+$ containing the cone Q of concave functions. Let $\Phi : \mathcal{Q} \to [0, +\infty]$ be a functional such that

(a) Φ is subadditive and positively homogeneous on $\mathcal{Q}$, that is

$$\Phi(f+g) \leq \Phi(f) + \Phi(g), \quad \Phi(\lambda f) = \lambda \Phi(f)$$

for all $f, g \in \mathcal{Q}$ and $\lambda \geq 0$,
(b) Φ is monotone on $\mathcal{Q}$, that is, if $g, f \in \mathcal{Q}$ and $g(t) \leq f(t)$ for all $t > 0$, then

$$\Phi(g) \leq \Phi(f)$$

(c) Φ is non-trivial in the sense that, if $\Delta(t) = \min(1, t), t > 0$, then $0 < \Phi(\Delta) < \infty$.

Any such functional Φ will be called a *K-interpolation functional* on $\mathcal{Q}$.

Let $\mathbf{E} = (E_0, E_1)$ be a Banach couple and Φ be a K-interpolation functional on Q. We set

$$\mathbf{E}_\Phi \;=\; \{x \in S(\mathbf{E}) : \Phi(K(\cdot, x)) < \infty\}.$$

Equipped with the norm

$$\|x\|_\Phi \;=\; \Phi(K(\cdot, x)), \quad x \in \mathbf{E}_\Phi,$$

the space $\mathbf{E}_\Phi$ is an interpolation space with respect to the couple $\mathbf{E}$ with constant 1. We recall that a normed space E, $I(\mathbf{E}) \subset E \subset S(\mathbf{E})$, is called an *interpolation space* with respect to $\mathbf{E}$ with constant C, if every bounded operator $A : S(\mathbf{E}) \to S(\mathbf{E})$ such that for each $i = 0, 1$ the restriction of A to E_i maps E_i into E_i, acts boundedly on E and $\|A\|_{E\to E} \leq C \max(\|A\|_{E_0 \to E_0}, \|A\|_{E_1 \to E_1})$. We shall refer to the space $(\mathbf{E}_\Phi, \|\cdot\|_\Phi)$ as a K-interpolation space.

We remark that the set $\overline{Q}$ of all (non-negative) quasiconcave functions on $\mathbb{R}^+$ is a cone which properly contains the cone Q. If Φ is a K-interpolation functional on the cone $\overline{Q}$ then it follows from earlier remarks that, for all $1 \leq p \leq \infty$,

$$\mathbf{E}_\Phi \;=\; \{x \in S(\mathbf{E}) : \Phi(K(\cdot, x)) < \infty\} = \{x \in S(\mathbf{E}) : \Phi(K_p(\cdot, x)) < \infty\}$$

and that the norm given by setting

$$\|x\|_\Phi^{(p)} \;=\; \Phi(K_p(\cdot, x)), \quad x \in \mathbf{E}_\Phi,$$

(which is an interpolation norm with respect to the couple $\mathbf{E}$ with constant 1) is equivalent to the norm $\|\cdot\|_\Phi$ on $\mathbf{E}_\Phi$. In this case, we will denote by $\mathbf{E}_\Phi^{(p)}$ the space $\mathbf{E}_\Phi$ equipped with the norm $\|\cdot\|_\Phi^{(p)}$.

We now introduce several properties for a K-interpolation functional $\Phi : \mathcal{Q} \to [0, \infty]$ related to order continuity and smoothness on the cone

$\mathcal{Q}$ relative to a given Banach couple $\mathbf{E}$. These properties bear an obvious similarity to corresponding order continuity and smoothness properties for the norms on Banach function spaces and we will show in the following section how these properties are automatically inherited from underlying parameter spaces used to define important classes of K-interpolation spaces.

Let Φ be a K-interpolation functional on the cone $\mathcal{Q}$. The functional Φ is said to be

(a) *order-continuous on* $\mathcal{Q}$, denoted $\Phi \in (A)$, if and only if

$$f, f_n \in \mathcal{Q},\ f_n(t) \to 0,\ \text{for all } t > 0,\ \Phi(f) < \infty \implies \Phi(\min(f, f_n)) \to 0.$$

(b) *lower semicontinuous on* $\mathcal{Q}$, denoted $\Phi \in (C)$, if and only if

$$f_n, f \in \mathcal{Q}, f_n(t) \to f(t) \text{ for all } t > 0,\ \Phi(f) < \infty \Longrightarrow \Phi(f) \leq \underline{\lim}_n \Phi(f_n).$$

(c) *strictly monotone* on the non-empty set $T \subseteq \mathbb{R}^+$, denoted $\Phi \in (SM|T)$, if and only if

$$f, g \in \mathcal{Q},\ f \leq g \text{ and } f \neq g \text{ on } T,\ \Phi(g) < \infty \implies \Phi(f) < \Phi(g).$$

We now consider certain properties for an interpolation functional Φ on $\mathcal{Q}$ which are of Kadec-Klee type. For convenience, we will refer to these as properties of type (H), with our terminology deriving from the practice of sometimes referring to the Kadec-Klee property in normed spaces as property (H).

The functional Φ is said to have *property* (H_0) on $\mathcal{Q}$ denoted $\Phi \in (H_0)$, if and only if

$$f_n, f \in \mathcal{Q},\ f_n(t) \to f(t),\ \text{for all } t > 0,\ \Phi(f) < \infty,$$
$$\Phi(f_n) \to_n \Phi(f) \Longrightarrow \Phi(|f_n - f|^\wedge) \to_n 0,$$

where $h^\wedge$ denotes the least concave majorant of h.

If the domain $\mathcal{Q}$ of the functional Φ contains all functions of the form $|f - g|$, where $f, g \in \mathcal{Q}$, then Φ is said to have *property* (H_1), denoted $\Phi \in (H_1)$, if and only if

$$f_n, f \in \mathcal{Q},\ f_n(t) \to f(t),\ \text{for all } t > 0,\ \Phi(f) < \infty,$$
$$\Phi(f_n) \to_n \Phi(f) \Longrightarrow \Phi(|f_n - f|) \to_n 0.$$

Theorem 4.1. *Let* $\mathbf{E} = (E_0, E_1)$ *be a Banach couple, let* $\mathcal{T}$ *be a Hausdorff linear topology on* $S(\mathbf{E})$ *weaker than the norm topology and let* Φ *be a* K*-interpolation functional on the cone* $\mathcal{Q} \supseteq Q$. *If* $\emptyset \neq T \subseteq (0, \infty)$ *and*

(i) $\mathbf{E} \in (D_{\mathcal{T}}|\mathbf{E}_\Phi)$ *relative to* T, $\mathbf{E} \in (C_{\mathcal{T}}|\mathbf{E}_\Phi)$ *relative to* $(0,\infty)$;
(ii) $\Phi \in (A)$, (C) *and* $(SM|T)$;
(iii) $\Phi \in (H_0)$ *or* $\Phi \in (H_1)$,

then $\mathbf{E}_\Phi$ *has the Kadec-Klee property with respect to* $\mathcal{T}$.

Proof. Let $x_n, x \in \mathbf{E}_\Phi, x \neq 0, x_n \to_{\mathcal{T}} x$ and $\|x_n\|_\Phi \to_n \|x\|_\Phi$. We first show that

$$x_n \longrightarrow_{S(\mathbf{E})} x.$$

Suppose this is not true. Since $\mathbf{E} \in (D_\tau|\mathbf{E}_\Phi)$, there exists $t_0 \in T$ such that $K(t_0, x_n) \not\to K(t_0, x)$. By the assumption $\mathbf{E} \in (C_{\mathcal{T}}|\mathbf{E}_\Phi)$ relative to $\mathbb{R}^+$, it follows that

$$K(t,x) \le \underline{\lim}_n K(t, x_n), \quad \text{for all } t > 0, \tag{4.1}$$

hence there exists $\epsilon > 0$ and a subsequence, denoted again by $\{x_n\}$, such that

$$\lim_n K(t_0, x_n) \ \ge\ K(t_0, x) + \epsilon. \tag{4.2}$$

Denote by f the smallest function in Q such that

$$f(t) \ge K(t,x), \text{ for all } t > 0 \quad \text{and} \quad f(t_0) = K(t_0,x) + \epsilon.$$

Then $\Phi \in (SM|T)$ implies $\Phi(K(\cdot,x)) < \Phi(f)$. Moreover, if

$$\alpha(t) = \begin{cases} 1, & \text{if } t \ge t_0; \\ \dfrac{t}{t_0}, & \text{if } 0 < t \le t_0, \end{cases}$$

then

$$f(t) \le K(t,x) + \epsilon\alpha(t) \le \left(1 + \frac{\epsilon}{K(t_0,x)}\right) K(t,x),$$

for all $t > 0$, and consequently $\Phi(f) < \infty$.

If $n = 1, 2, \ldots$, define

$$f_n(t) = \min(f(t), K(t, x_n)),$$

for all $t > 0$. The inequalities (4.1) and (4.2) imply $f_n(t) \to_n f(t)$ for all $t > 0$. Since $\Phi \in (C)$, it follows that

$$\Phi(f) \ \le\ \underline{\lim}_n \Phi(f_n).$$

Now

$$\begin{aligned}\|x\|_\Phi &= \Phi(K(\cdot,x)) < \Phi(f) \le \underline{\lim}_n \Phi(f_n) \le \underline{\lim}_n \Phi(K(\cdot,x_n))\\ &= \underline{\lim}_n \|x_n\|_\Phi = \|x\|_\Phi\end{aligned}$$

which is a contradiction. Appealing to Proposition 2.7, we have

$$x_n \longrightarrow_{S(\mathbf{E})} x.$$

Consequently,

$$K(t, x_n - x) \longrightarrow_n 0, \quad \text{for all } t > 0, \tag{4.3}$$

and so also

$$K(t, x_n) \longrightarrow_n K(t,x), \quad \text{for all } t > 0. \tag{4.4}$$

Now (4.3) and $\Phi \in (A)$ imply that

$$\Phi(\ \min(K(\cdot, x_n - x), 2K(\cdot,x))\ ) \longrightarrow_n 0; \tag{4.5}$$

and (4.4) and $\Phi \in (H_0)$ or (H_1) on Q imply that

$$\Phi(|K(\cdot,x_n) - K(\cdot,x)|^\wedge) \to_n 0 \quad \text{or} \quad \Phi(|K(\cdot,x_n) - K(\cdot,x)|) \to_n 0. \tag{4.6}$$

Since for each $t > 0$, $K(t,\cdot)$ is a norm, we have

$$K(t, x_n - x) \le |K(t,x_n) - K(t,x)| + \min\left(K(t,x_n - x), 2K(t,x)\right), \tag{4.7}$$

for all $t > 0$, and therefore it follows from (4.5), (4.6) and (4.7) that

$$\|x_n - x\|_\Phi = \Phi(K(\cdot, x_n - x)) \longrightarrow_n 0,$$

which completes the proof of the theorem. □

Banach couple and τ be a Hausdorff linear K-interpolation functional with the properties $(A|\mathbf{E})$, $(C|\mathbf{E})$, $(SM|T)$ and either (H_0) or (H_1). If both E_0 and E_1 have the Kadec-Klee property with respect to τ then so does the interpolation space $(\mathbf{E}_\Phi, \|\cdot\|_\Phi)$.

We remark that if Φ is defined on a cone $\mathcal{Q}$ containing the cone $\overline{Q}$ of quasiconcave functions then the proof of the preceding theorem may be simplified by replacing the function f considered in the proof by the function F defined by setting

$$F(t) = \max\left(K(t,x), \min\left\{K(t_0,x) + \epsilon, \frac{K(t_0,x)+\epsilon}{t_0} t\right\}\right), \quad t > 0.$$

We now introduce two properties of a K-interpolation functional Φ on the cone $\mathcal{Q}$ which will play a similar role in the study of interpolation of local uniform convexity to that played by the properties $(H_0), (H_1)$ in the corresponding study of Kadec-Klee properties.

The K-interpolation functional Φ on the cone $\mathcal{Q}$ is said to have property (LUR_0) relative to the non-empty set $T \subseteq (0,\infty)$, denoted $\Phi \in (LUR_0|T)$ if and only if whenever $f_n, f \in \mathcal{Q}$ satisfy

$$\Phi(f_n) \to_n \Phi(f),\ \Phi(f_n + f) \to_n 2\Phi(f) < \infty,$$

it follows that

$$\Phi(|f_n - f|^\wedge) \to_n 0,\ \text{and}\ f_n(t) \to f(t)\ \text{for all}\ t \in T.$$

If the domain $\mathcal{Q}$ of the functional Φ contains all functions of the form $|f - g|$, for all $f, g \in \mathcal{Q}$, then Φ is said to have property (LUR_1) relative to the non-empty set $T \subseteq (0,\infty)$, denoted $\Phi \in (LUR_1|T)$ if and only if whenever $f_n, f \in \mathcal{Q}$ satisfy

$$\Phi(f_n) \to_n \Phi(f),\ \Phi(f_n + f) \to_n 2\Phi(f) < \infty,$$

it follows that

$$\Phi(|f_n - f|) \to_n 0,\ \text{and}\ f_n(t) \to f(t)\ \text{for all}\ t \in T.$$

We remark that the local uniform convexity is occasionally referred to by some authors as local uniformly rotundity or (LUR). This is the source of the above terminology.

Theorem 4.2. *Let* $\mathbf{E} = (E_0, E_1)$ *be a Banach couple, and let* Φ *be a* K*-interpolation functional on the cone* $\mathcal{Q} \supseteq Q$. *If* $\phi \neq T \subseteq (0,\infty)$ *and if*
(i) $\mathbf{E} \in (DGL|\mathbf{E}_\Phi)$ *relative to* T,
(ii) $\Phi \in (A)$ and $\Phi \in (SM|T)$,
(iii) $\Phi \in (LUR_0|T)$ or $\Phi \in (LUR_1|T)$,
then $\mathbf{E}_\Phi$ *is locally uniformly convex.*

Proof. Let $x_n, x \in \mathbf{E}_\Phi, n = 1, 2, \ldots,$ satisfy $\|x_n\|_\Phi \to_n \|x\|_\Phi$ and $\|x_n + x\|_\Phi \to_n 2\|x\|_\Phi$. Since

$$\|x_n+x\|_\Phi \le \Phi(K(\cdot,x_n)+K(\cdot,x)) \le \Phi(K(\cdot,x_n))+\Phi(K(\cdot,x)) = \|x_n\|_\Phi+\|x\|_\Phi,$$

for all $n = 1, 2, \ldots,$ we have

$$\Phi(K(\cdot,x_n) + K(\cdot,x)) \ \longrightarrow_n\ 2\Phi(K(\cdot,x)).$$

Consequently, by the assumption that $\Phi \in (LUR_0|T)$ or $\Phi \in (LUR_1|T)$, it follows that

(4.8)
$$\Phi(|K(\cdot,x_n) - K(\cdot,x)|^\wedge) \ \longrightarrow_n \ 0 \quad \text{or} \quad \Phi(|K(\cdot,x_n) - K(\cdot,x)|) \ \longrightarrow_n \ 0,$$

and

(4.9)
$$K(t,x_n) \ \longrightarrow_n \ K(t,x), \quad \text{for all } t \in T.$$

For each $n = 1, 2, \ldots$, define

$$M_n(t) \ = \ \min(K(t,x_n + x), 2K(t,x)), \quad \text{for } t > 0.$$

Then

$$\begin{aligned} K(t,x_n + x) \quad &= \quad M_n(t) + (K(t,x_n + x) - 2K(t,x))^+ \\ &\leq \quad M_n(t) + |K(t,x_n) - K(t,x)|, \quad \text{for } t > 0, \end{aligned}$$

and hence by (4.8),

$$\lim_n \Phi(K(\cdot,x_n + x)) \ \leq \ \underline{\lim}_n \Phi(M_n).$$

On the other hand,

$$M_n(t) \ \leq \ K(t,x_n + x), \quad \text{for all } t > 0,$$

hence

(4.10)
$$\lim_n \Phi(K(\cdot,x_n + x)) \ = \ \lim_n \Phi(M_n).$$

We will next show that

(4.11)
$$K(t,x_n + x) \ \longrightarrow_n \ 2K(t,x), \quad \text{for all } t \in T.$$

Since

$$\overline{\lim}_n K(t,x_n + x) \ \leq \ \overline{\lim}_n \,(K(t,x_n) + K(t,x)) \ = \ 2K(t,x), \quad \text{for all } t > 0,$$

it suffices to show

$$2K(t,x) \ \leq \ \underline{\lim}_n K(t,x_n + x), \quad \text{for all } t \in T.$$

If this is not true, then there exists $t_0 \in T$ such that

$$2K(t_0,x) \ > \ \underline{\lim}_n K(t_0,x_n + x).$$

Passing to a subsequence and relabelling if necessary, we may assume that there exists $\epsilon > 0$ and a natural number $N \in \mathbb{N}$ such that

$$K(t_0, x_n + x) \;\leq\; 2K(t_0, x) - 2\epsilon,$$

and consequently

$$M_n(t_0) \;\leq\; 2K(t_0, x) - 2\epsilon,$$

for all $n \geq N$. Write $Y_0 = 2K(t_0, x) - 2\epsilon$. Since the functions M_n are concave and increasing, we may assume that the sequence D_n of left derivatives of the functions M_n at the point t_0, $n = 1, 2, \ldots$, has limit D. If $\delta > 0$ is such that $0 < \delta t_0 < \epsilon/2$, then for all $t > 0$,

$$(D - \delta)(t - t_0) + Y_0 \;<\; (D + \delta)(t - t_0) + Y_0 + \epsilon.$$

Define

$$\psi(t) \;=\; \min\left(2K(t, x), (D + \delta)(t - t_0) + Y_0 + \epsilon\right), \quad t > 0,$$

then

$$\psi(t_0) \;<\; 2K(t_0, x), \text{and} \qquad \psi(t) \;\leq\; 2K(t, x), \tag{4.12}$$

for all $t > 0$. We may assume that $|D_n - D| < \delta$ for all $n \geq N$. Consequently,

$$\begin{aligned} M_n(t) &\leq D_n(t - t_0) + M_n(t_0) \\ &\leq \max\left((D - \delta)(t - t_0), (D + \delta)(t - t_0)\right) + Y_0 \\ &< (D + \delta)(t - t_0) + Y_0 + \epsilon \end{aligned}$$

for all $t > 0$. Using the definition of M_n, it follows that

$$M_n(t) \leq \psi(t) \leq 2K(t, x) \tag{4.13}$$

for all $t > 0$ and $n \geq N$. From (4.10), (4.12), (4.13) and the assumption that $\Phi \in (SM|T)$, we have

$$2\|x\|_\Phi = \lim_k \Phi(K(\cdot, x_n + x)) = \lim_n \Phi(M_n) \leq \Phi(\psi) < 2\Phi(K(\cdot, x)) = 2\|x\|_\Phi,$$

which is a contradiction. Thus (4.11) holds. The assertions (4.9), (4.11) and the assumption $\mathbf{E} \in (DGL|\mathbf{E}_\Phi)$ relative to T imply

$$x_n - x \;\longrightarrow_{S(\mathbf{E})}\; 0.$$

It follows that $K(t, x_n - x) \to 0$, for all $t > 0$, and therefore the assumption $\Phi \in (A)$ implies

$$\Phi(\min(K(\cdot, x_n - x), 2K(\cdot, x))) \;\longrightarrow_n\; 0. \tag{4.14}$$

It now follows from (4.7), (4.8) and (4.14) that

$$\Phi(K(\cdot, x_n - x)) \longrightarrow_n 0,$$

which completes the proof of the theorem. □

Theorem 4.3. *Let* $\mathbf{E} = (E_0, E_1)$ *be a Banach couple, and let* Φ *be a* K*-interpolation functional on a cone* $\mathcal{Q}$ *containing the cone* $\overline{Q}$ *of quasiconcave functions. If* $\emptyset \neq T \subseteq (0, \infty)$ *and if*

(i) $\mathbf{E} \in (DGL^p | \mathbf{E}_\Phi)$ relative to T for some $1 \leq p < \infty$,

(ii) $\Phi \in (A)$ and $\Phi \in (SM|T)$,

(iii) $\Phi \in (LUR_0|T)$ or $(LUR_1|T)$,

then $\mathbf{E}_\Phi^{(p)}$ is locally uniformly convex.

Proof. The proof follows exactly the same lines as that of the previous theorem with $K(\cdot,\cdot)$ replaced by $K_p(\cdot,\cdot)$. However, the argument on which the estimate (4.13) is based is no longer valid. This difficulty is circumvented by replacing the concave function ψ by the quasi-concave function φ where

$$\varphi(t) = \min\left(2K_p(t,x), \max\left\{2K_p(t_0,x) - \epsilon, \frac{2K_p(t_0,x) - \epsilon}{t_0} t\right\}\right),$$

for all $t > 0$. One then obtains directly the counterpart to (4.13) in the form

$$M_n(t) \leq \varphi(t) \leq 2K_p(t,x).$$

The remaining arguments remain the same, and this suffices to complete the proof of the Theorem. □

5. The K-Interpolation Functionals $\Phi_{\rho,F}$ and Φ_G

We begin by discussing two basic methods of constructing K-interpolation functionals. We suppose first that F is a Banach function space on (T, μ) and let $\rho(t)$, $t \in T$ be a positive weight. Let $\mathcal{Q}$ be the cone generated by all functions of the form $h = |f - g|$, where $f, g \in \overline{Q}$. We define the functional $\Phi_{\rho,F}$ on the cone $\mathcal{Q}$ by setting

$$\Phi_{\rho,F}(f) = \|f \; \rho \; \chi_T\|_F, \quad f \in \mathcal{Q},$$

where χ_D is the indicator function of the measurable set D. If $0 < \|\Delta \; \rho \; \chi_T\|_F < \infty$, then $\Phi_{\rho,F}$ is a K-interpolation functional on $\mathcal{Q}$. If $\mathbf{E}$

is a Banach couple, we set

$$\mathbf{E}_{\rho,F} = \mathbf{E}_{\Phi_{\rho,F}}$$

for notational simplicity and observe that $\mathbf{E}_{\rho,F}$ is a Banach space. We observe that, for every Banach couple $\mathbf{E} = (E_0, E_1)$, each interpolation space of the form $E_0 + tE_1$, $t > 0$ arises as a very special case of this construction. In fact, we need only take $T = \{t\}$, $\rho(t) = 1$, $\mu(\{t\}) = 1$, and $F = \mathbb{C}$.

The general construction above yields, as an important special case, the real interpolation spaces $\mathbf{E}_{\theta,p}$, $0 < \theta < 1$, $1 \leq p < \infty$, (see [5]) consisting of all $x \in S(\mathbf{E})$ for which the functional

$$\|x\|_{\theta,p} = \left\{ \int_0^\infty [t^{-\theta} K(t, x; \mathbf{E})]^p \frac{dt}{t} \right\}^{1/p}$$

is finite. Here $T = [0, \infty)$, $F = L^p([0, \infty))$ and the weight function ρ is given by setting $\rho(t) = t^{-\theta - 1/p}$ for all $t > 0$.

For each $1 < p < \infty$, the interpolation functional $\Phi_{\rho,F}$ yields the K-interpolation spaces $\mathbf{E}^{(p)}_{\rho,F} := \mathbf{E}^{(p)}_{\Phi_{\rho,F}}$ consisting of all $x \in S(\mathbf{E})$ for which

$$\|x\|^{(p)}_{\Phi_{\rho,F}} := \Phi_{\rho,F}(K_p(\cdot, x)) < \infty$$

Of particular interest is the special case given by the (so-called) "discrete K-method" (see [24]). Let Y be a Banach space with a normalized unconditional basis $\{e_n\}_{n=1}^\infty$ whose unconditionality constant is one and let $\{a_n\}_{n=1}^\infty$ and $\{b_n\}_{n=1}^\infty$ be sequences of positive numbers for which $\sum_{n=1}^\infty \min(a_n, b_n) < \infty$. If $1 \leq p \leq \infty$, the space $\mathcal{K}_p(\mathbf{E}, \{a_n\}, \{b_n\}, Y)$ is defined to be the set of all elements $x \in S(\mathbf{E})$ such that $\sum_{n=1}^\infty k_p(x, a_n, b_n) e_n$ converges in Y, normed by setting

$$\|x\|^{(p)} := \|\sum_{n=1}^\infty k_p(x, a_n, b_n) e_n\|_Y$$

where

$$k_p(x, a_n, b_n) := \inf\{\left(a_n^p \|x^0\|_0^p + b_n^p \|x^1\|_1^p\right)^{1/p} : x = x^0 + x^1, x_i \in E_i, i = 0, 1\}.$$

For every such choice of unconditional basis $\{e_n\}_{n=1}^\infty$ and sequences $\{a_n\}_{n=1}^\infty$ and $\{b_n\}_{n=1}^\infty$, it follows from [24], Proposition 2.g.4, that the space $\mathcal{K}_p(\mathbf{E}, \{a_n\}, \{b_n\}, Y)$ is an interpolation space for the couple $\mathbf{E}$ with interpolation constant 1. In what follows, it will always be assumed that

$$\frac{b_n}{a_n} \neq \frac{b_m}{a_m},$$

whenever $n \neq m,\ n, m = 1, 2, \ldots$. We set $t_n = b_n/a_n$, and define the weight function ρ on $T := \{t_n : n = 1, 2, \ldots\}$ by setting $\rho(t_n) = a_n,\ n = 1, 2, \ldots$. If now $x \in \mathcal{K}_p(\mathbf{E}, \{a_n\}, \{b_n\}, Y)$, observe that

$$k_p(x, a_n, b_n) = a_n K_p(\frac{b_n}{a_n}, x) = \rho(t_n) K_p(t_n, x)$$

for all $n = 1, 2, \ldots$. From this it follows that

$$\|x\|^{(p)} = \|\sum_{n=1}^{\infty} k_p(x, a_n, b_n) e_n\|_Y = \|K_p(\cdot, x)\rho(\cdot)\chi_T(\cdot)\|_F$$

where F is the Banach function space induced by Y on the measure space (T, μ), where μ denotes counting measure.

We now turn to the second basic method for constructing interpolation functionals. If G is a fully symmetric space on $[0, \alpha),\ 0 < \alpha \leq \infty$, we define the K-interpolation functional Φ_G on Q by setting

$$\Phi_G(f) = \begin{cases} \|f'\|_G, & \text{if } f \in Q_0, f' \in G; \\ \infty, & \text{otherwise.} \end{cases}$$

for all $f \in Q$. See, for example, [4] Definition 5.1.18. It is easily seen that Φ_G is a K-interpolation functional. Although the K-interpolation functional Φ_G is formally defined on the subcone $\overline{Q}_0$ of quasiconcave functions which vanish at 0, the functional Φ_G is, in general, neither subadditive nor monotone on $\overline{Q}_0$, in contrast to interpolation functionals of the form $\Phi_{\rho,F}$. Nevertheless, let us note that on the cone Q_0, every real interpolation functional Φ coincides with an interpolation functional of the form Φ_G, for some fully symmetric space G on $\mathbb{R}^+$. In fact, if $\mathbf{E} = (L_1[0, \infty), L_\infty[0, \infty))$ and if $G = \mathbf{E}_\Phi$, then G is fully symmetric [21] and if $f \in Q_0$, then $f(\cdot) = K(\cdot, f') = K(\cdot, f'; \mathbf{E})$ and so

$$\Phi(f) = \Phi(K(\cdot, f')) = \|f'\|_{\mathbf{E}_\Phi} = \|f'\|_G = \Phi_G(f).$$

In the sequel, we shall denote the space $\mathbf{E}_{\Phi_G}$ by $\mathbf{E}_G$ and its norm by $\|\cdot\|_{\mathbf{E}_G}$. We remark that if $\mathcal{C}_1, \mathcal{C}_\infty$ denote the ideals of trace class and compact operators respectively on some separable Hilbert space, if $\mathbf{E}$ is the Banach couple $(\mathcal{C}_1, \mathcal{C}_\infty)$, and if $G = L_p[0, \infty), 1 < p < \infty$, then the interpolation space $\mathbf{E}_G$ is the (non-commutative) Schatten ideal $\mathcal{C}_p$ consisting of all operators x for which the sequence $\{s_n(x)\}_{n=1}^{\infty}$ of singular values of x is an element of l^p with norm given by

$$\|x\|_{\mathcal{C}_p} = \|\{s_n(x)\}_{n=1}^{\infty}\|_{l^p}.$$

More generally, if $(\mathcal{M}, \tau)$ is a semi-finite von Neumann algebra on some Hilbert space, with predual $\mathcal{M}_*$, if $\mathbf{E}$ is the Banach couple $(\mathcal{M}_*, \mathcal{M})$ and if G is a fully symmetric Banach function space on $\mathbb{R}^+$, then the interpolation space $\mathbf{E}_G$ is the non-commutative space $G(\mathcal{M}, \tau)$ corresponding to G as given in, for example, [9]. More detailed considerations of these spaces will be deferred to [12].

In the case of the K-interpolation functionals $\Phi_{\rho,F}$, Φ_G, the theorem which follows shows that the continuity and smoothness properties which occur in the principal interpolation theorems of the preceding section may be readily inferred from more familiar corresponding properties of the parameter spaces F, G.

Theorem 5.1. (i) *Let $(F, \|\cdot\|_F)$ be a Banach function space on the measure space (T, μ) where T is either the interval $(0, \alpha)$ for some $0 < \alpha \leq \infty$ and μ is Lebesgue measure m, or $T \subseteq \mathbb{R}^+$ is a countable subset $\{t_n\}_{n\in\mathbb{N}}$ and μ is counting measure. Let ρ be a strictly positive weight on T.*

(a) *If F has order continuous norm, then $\Phi_{\rho,F} \in (A)$.*

(b) *If the norm on F is Fatou, then $\Phi_{\rho,F} \in (C)$.*

(c) *If the norm on F is strictly monotone then $\Phi_{\rho,F} \in (SM|T)$.*

(d) *If F has the Kadec-Klee property for local convergence in measure then $\Phi_{\rho,F} \in (H_1)$.*

(e) *If F is locally uniformly convex then $\Phi_{\rho,F} \in (LUR_1|T)$.*

(ii) *Let G be a fully symmetric Banach function space on $T = (0, \alpha)$, $0 < \alpha \leq \infty$ with respect to Lebesgue measure m.*

(a) *If the norm on G is Fatou then $\Phi_G \in (C)$.*

(b) *If the norm on G is strictly K-monotone then $\Phi_G \in (SM|T)$.*

(c) *If G has the Kadec-Klee property for local convergence in measure then $\Phi_G \in (H_0)$.*

(d) *If G is locally uniformly convex then $\Phi_G \in (LUR_0|T)$.*

Proof. (i) The proof of each of the assertions (a)-(d) is routine verification and the details are omitted.

To see that (e) holds, suppose that F is locally uniformly convex. We suppose that $f_n, f \in \overline{Q}$, that $\Phi_{\rho,F}(f_n) \to_n \Phi_{\rho,F}(f)$, and that $\Phi_{\rho,F}(f_n + f) \to_n 2\Phi_{\rho,F}(f) < \infty$. Since F is locally uniformly convex, it follows immediately that $f_n \rho \chi_T \to_F f \rho \chi_T$. It follows further that $f_n \to f$ locally in μ-measure. From Corollary 1.11, it follows that $f_n(t) \to f(t)$ for all $t \in T$ and so $\Phi_{\rho,F} \in (LUR_1|T)$.

(ii) (a) Assume that the norm on G is Fatou. We suppose that $f_n, f \in Q,\ n = 1, 2, \ldots$ satisfy $\Phi_G(f) < \infty$ and that $f_n(t) \to f(t)$ for all $t > 0$. We may assume further that $\underline{\lim}_n \Phi_G(f_n) < \infty$, otherwise there is nothing to prove. Consequently, by passing to a subsequence and relabelling if necessary, we may assume that $f_n, f \in Q_0,\ n = 1, 2, \ldots,$ and that $f_n', f' \in G,\ n = 1, 2, \ldots,$ From Proposition 1.9, it follows that $f_n' \to f'$ locally in measure. By lower-semicontinuity of the norm on G, it follows that

$$\Phi_G(f) = \|f'\|_G \leq \underline{\lim}_n \|f_n'\|_G = \underline{\lim}_n \Phi_G(f_n),$$

and this implies that $\Phi_G \in (C)$.

(b) Let $f, g \in Q$ and suppose that $f \leq g,\ f \neq g$ on T and that $\Phi_G(g) < \infty$. It follows immediately that $f' \prec g'$ and that $f' \neq g'$. Consequently, if the norm on G is strictly K-monotone and if $g' \in G$ then

$$\Phi_G(f) = \|f'\|_G < \|g'\|_G = \Phi_G(g)$$

and this implies that $\Phi_G \in (SM|T)$.

(c) Assume that G has the Kadec-Klee property for local convergence in measure, that $f_n, f \in Q,\ f_n(t) \to f(t)$ for all $t > 0$, and that $\Phi_G(f_n) \to_n \Phi_G(f) < \infty$. It follows that $f \in Q_0$ and consequently we may assume that $f_n \in Q_0$ for all $n = 1, 2, \ldots.$ From Proposition 1.9, it follows that $f_n' \to_{lm} f'$. Since

$$\|f_n'\|_G = \Phi_G(f_n) \to_n \Phi_G(f) = \|f'\|_G$$

and since G has the Kadec-Klee property for local convergence in measure, it follows that $f_n' \to f'$ in G. By a well-known property of rearrangements (see [21]), we have that

$$|f_n(t) - f(t)| = \left| \int_0^t f_n'(s)ds - \int_0^t f'(s)ds \right| \leq \int_0^t (f_n' - f')^*(s)ds,$$

for each $t > 0$. Since the function $t \to \int_0^t (f_n' - f')^*(s)ds$ is concave, it follows that

$$|f_n - f|^\wedge(t) \leq \int_0^t (f_n' - f')^*(s)ds$$

for all $t > 0$ and consequently

$$(|f_n - f|^\wedge)' \prec (f_n' - f')^*.$$

Since G is fully symmetric, it now follows that

$$\Phi_G(|f_n - f|^\wedge) \leq \Phi_G\left(\int_0^t (f_n' - f')^*(s)ds\right) = \|f_n' - f'\|_G \to 0,$$

and this implies that $\Phi_G \in (H_0)$.
(d) Assume that G is locally uniformly convex. Let $f_n, f \in Q$, $n = 1, 2, \ldots$ and suppose that $\Phi(f_n) \to \Phi(f)$ and $\Phi(f_n + f) \to 2\Phi(f) < \infty$. Since $\Phi(f) < \infty$, we may assume that $f_n, f \in Q_0$ and that $f_n', f' \in G$ for all $n = 1, 2, \ldots$. Since

$$\|f_n'\|_G = \Phi(f_n) \to \Phi(f) = \|f'\|_G$$

and

$$\|f_n' + f'\|_G = \Phi(f_n + f) \to 2\Phi(f) = 2\|f'\|_G$$

it follows from the local uniform convexity of G that

$$\|(f_n' - f')^*\|_G = \|f_n' - f'\|_G \to 0.$$

Finally, if G' denotes the symmetric Banach function space associated with G (see [21], II.4.6), and if we note that

$$|f_n - f|^\wedge(t) \leq \int_0^t (f_n' - f')^*(s)ds \leq \|f_n' - f'\|_G \|\chi_{[0,t)}\|_{G'}$$

then it now follows that $f_n(t) \to f(t)$ for every $t \in T = (0, \alpha)$ and so $\Phi_G \in (LUR_0|T)$. □

Remark. In contrast to the statement of Theorem 5.1(i)(a) preceding, if G is a symmetric function space with order continuous norm, then it does not follow that $\Phi_G \in (A)$. By way of example, we take G to be $L_1[0, \infty)$. In particular, G has order continuous norm. If $f_n(t) := \min(1, t/n)$, $n = 1, 2, \ldots$, and $f(t) = \min(1, t) = f_1(t)$ for all $t > 0$, then $f_n(t) \downarrow 0$ and $\min(f(t), f_n(t)) = f_n(t)$ for all $t > 0$. However,

$$\Phi_G(\min(f, f_n)) = \Phi_G(f_n) = \|f_n'\|_1 = 1$$

for all $n = 1, 2, \ldots$ and from this it follows that $\Phi_G \notin (A)$.

Theorem 5.2. *Let G be a fully symmetric Banach function space on $[0, \alpha)$, $0 < \alpha \leq \infty$ with order continuous norm.*

(i) *If $\alpha < \infty$, then $\Phi_G \in (A)$.*
(ii) *If $\alpha = \infty$ and if $G \not\subseteq L_1[0, \infty)$, then $\Phi_G \in (A)$.*

As the proof of the preceding Theorem is technical, we prefer to present the details in Appendix A in order to focus on the application of the ideas contained in the general interpolation theorems of the previous section to the concrete case of the functionals $\Phi_{\rho,F}$, Φ_G.

Theorem 5.3. *Let $(F, \|\cdot\|_F)$ be a Banach function space on the measure space (T, μ) where T is either the interval $(0, \alpha)$ for some $0 < \alpha \leq \infty$ and μ*

is Lebesgue measure m*, or* T *is a countable subset* $\{t_n\}_{n\in\mathbb{N}}$ *and* μ *is counting measure, and let* ρ *be a strictly positive weight on* T*. Let* $\mathbf{E}$ *be a Banach couple and let* $\mathcal{T}$ *be a Hausdorff linear topology on* $S(\mathbf{E})$ *weaker than the norm topology. If*

(i) F *has the Kadec-Klee property for local convergence in measure and has strictly monotone norm,*
(ii) $\mathbf{E} \in (D_{\mathcal{T}}|\mathbf{E}_{\rho,F})$ *relative to* T *and* $\mathbf{E} \in (C_{\mathcal{T}}|\mathbf{E}_{\rho,F})$ *relative to* $\mathbb{R}^+$*,*

then $\mathbf{E}_{\rho,F}$ *has the Kadec-Klee property with respect to* $\mathcal{T}$*.*

The proof of the Theorem follows immediately from Proposition 2.6, Theorem 4.1 and Theorem 5.1(i).

Corollary 5.4. *Let* $(F, \|\cdot\|_F)$ *be a Banach function space on the measure space* (T, μ) *where* T *is either the interval* $(0, \alpha)$ *for some* $0 < \alpha \leq \infty$ *and* μ *is Lebesgue measure* m*, or* T *is a countable subset* $\{t_n\}_{n\in\mathbb{N}}$ *and* μ *is counting measure. Let* ρ *be a strictly positive weight on* T*. Suppose that* F *has the Kadec-Klee property for local convergence in measure and has strictly monotone norm. let* $\mathbf{E}$ *be a Banach couple and let* $\mathcal{T}$ *be a Hausdorff linear topology on* $S(\mathbf{E})$ *weaker than the norm topology. Assume that* $\mathbf{E}$ *is* $\mathcal{T}$*-closed. If* E_i *has the Kadec-Klee property with respect to* $\mathcal{T}$ *for each* $i = 0, 1$ *then* $\mathbf{E}_{\rho,F}$ *has the Kadec-Klee property with respect to* $\mathcal{T}$*.*

The proof follows directly from Theorem 5.3 via Proposition 2.6, Proposition 3.4 and Theorem 3.6.

It is worthwhile remarking that the preceding Corollary 5.4 applies to the special case given by the interpolation space $\mathcal{K}(\mathbf{E}, a_n, b_n, Y)$, where Y is a space with unconditional basis $\{e_n\}_{n=1}^{\infty}$ with unconditionality constant 1 having strictly monotone norm and having the Kadec-Klee property with respect to the weak topology $\mathcal{S}$ induced by the sequence of biorthogonal functionals associated with the basis $\{e_n\}_{n=1}^{\infty}$. We note that if the basis $\{e_n\}_{n=1}^{\infty}$ is 1−symmetric, then strict monotonicity of the norm on Y is implied by the Kadec-Klee property with respect to $\mathcal{S}$. See [1], [Si].

We consider next the interpolation of Kadec-Klee properties in interpolation spaces of the form $\mathbf{E}_G$. Before proceeding, we require some additional terminology. We recall (see [4], Chapter 5.1) that the space $E_0 + \infty E_1$ (known as the *Gagliardo completion of* E_0) consists of all $x \in S(\mathbf{E})$ for which the norm

$$K^{\infty}(x) = \sup_{0<t<\infty} K(t,x) = \lim_{t\to\infty} K(t,x)$$

is finite. We remark that if the unit ball of E_0 is sequentially closed in $S(\mathbf{E})$, then $E_0 + \infty E_1 = E_0$. See [4], Chapter 5, Theorem 1.4.

Definition 5.5. A Banach couple $\mathbf{E}$ will be said to have property (D^∞), written $\mathbf{E} \in (D^\infty)$ if and only if whenever $x_n, x \in S(\mathbf{E})$, $n = 1, 2, \ldots$, satisfy $x_n \to x \in S(\mathbf{E})$ and $K^\infty(x_n) \to K^\infty(x)$, it follows that

$$K^\infty(x_n - x) \to 0.$$

Theorem 5.6. *Let $\mathcal{T}$ be a Hausdorff linear topology on $S(\mathbf{E})$ weaker than the norm topology. Let G be a fully symmetric Banach function space on $(0,\alpha)$, $0 < \alpha \leq \infty$. Suppose that G has the Kadec-Klee property for local convergence in measure and has strictly K-monotone norm, and that $\mathbf{E} \in (D_{\mathcal{T}}|\mathbf{E}_G)$ with respect to $T = (0,\alpha)$, and $\mathbf{E} \in (C_{\mathcal{T}}|\mathbf{E}_G)$ with respect to $(0,\infty)$.*

(i) *If $0 < \alpha < \infty$, or, if $\alpha = \infty$ and $G \not\subseteq L_1[0,\infty)$, then $\mathbf{E}_G$ has the Kadec-Klee property with respect to $\mathcal{T}$.*
(ii) *If $\alpha = \infty$ and if $E_1 \subseteq E_0$ then $\mathbf{E}_G$ has the Kadec-Klee property with respect to $\mathcal{T}$.*
(iii) *If $\alpha = \infty$ and if $\mathbf{E} \in (D^\infty)$, then $\mathbf{E}_G$ has the Kadec-Klee property with respect to $\mathcal{T}$.*

Proof. Observe first that the assumption G has the Kadec-Klee property for local convergence in measure implies that the norm on G is a Fatou norm via Proposition 2.2; that G has order-continuous norm follows from [8], Proposition 2.1, and that $\Phi_G \in (H_0)$ follows from Theorem 5.1(ii)(c). Since the norm on G is Fatou and strictly K-monotone, it follows from Theorem 5.1(ii)(a), (b) respectively that $\Phi_G \in (C)$ and $\Phi_G \in (SM|T)$ with $T = (0,\alpha)$. If $\alpha < \infty$ or if $\alpha = \infty$ and $G \not\subseteq L_1[0,\infty)$, then it follows from the fact that the norm on G is order continuous and Theorem 5.2 that $\Phi_G \in (A)$ and the assertion of (i) now follows from Theorem 4.1.

(ii) Assume that $\alpha = \infty$ and that $E_1 \subseteq E_0$. Let $x_n, x \in \mathbf{E}_G$, $n = 1, 2, \ldots$, and suppose that $x_n \to_{\mathcal{T}} x$ and that $\|x_n\|_{\mathbf{E}_G} \to \|x_n\|_{\mathbf{E}_G}$. From the first part of the proof of Theorem 3.1, it follows that $x_n - x \to_{S(\mathbb{E})} 0$, and since $S(\mathbf{E}) = E_0 = E_0 + \infty E_1$, it follows further that $K^\infty(x_n - x) \to 0$. An inspection of the proof of Theorem 4.1 together with the remarks in the first part of (i) preceding now shows that the assertion of (ii) will follow provided it is shown that

$$\Phi_G(\min(K(\cdot, x_n - x), 2K(\cdot, x)) \to_n 0.$$

This follows by noting that since the norm on G is order continuous, the first part of the proof of Theorem 5.2 (see Appendix A) implies that

$$\lim_{s\to 0} \Phi_G(\min(2K(\cdot, x), s\mathbf{1}) = 0.$$

On the other hand, since $K^\infty(x_n - x) \to 0$, it is clear that, given $s > 0$,

$$K(\cdot, x_n - x) \le s\mathbf{1}$$

for all sufficiently large n, and this suffices to complete the proof of (ii).

(iii) We assume $\alpha = \infty$ and that $\mathbf{E} \in (D^\infty)$. In view of (i), it suffices to assume further that G embeds continuously into $L_1[0,\infty)$. Now suppose that $x_n, x \in \mathbf{E}_G$, $n = 1, 2, \ldots$, and suppose that $x_n \to_{\mathcal{T}} x$ and $\|x_n\|_{\mathbf{E}_G} \to \|x_n\|_{\mathbf{E}_G}$. Setting $f_n(t) = K(t, x_n)$, $f(t) = K(t, x)$ for all $t > 0$ and $n = 1, 2, \ldots$, it follows as in the proof of Theorem 4.1 that $x_n \to_{S(\mathbf{E})} x$ and $f_n(t) \to f(t)$ for every $t > 0$. From Proposition 2.9, it then follows that $f_n' \to f'$ m-almost everywhere and hence locally in measure. Since G has the Kadec-Klee property for local convergence in measure, it follows that $f_n' \to_G f'$ and hence $f_n' \to_{L_1} f'$. This implies that $K^\infty(x_n) \to K^\infty(x)$ and so from the assumption that $\mathbf{E} \in (D^\infty)$, it follows also that $K^\infty(x_n - x) \to 0$. That $\mathbf{E}_G$ has the Kadec-Klee property with respect to $\mathcal{T}$ now follows as in the final part of (ii) preceding. □

We now turn to the interpolation of local uniform convexity.

Theorem 5.7. *Let $(F, \|\cdot\|_F)$ be a Banach function space on the measure space (T, μ) where T is either the interval $(0, \alpha)$ for some $0 < \alpha \le \infty$ and μ is Lebesgue measure m, or T is a countable subset $\{t_n\}_{n \in \mathbb{N}}$ and μ is counting measure. Let ρ be a strictly positive weight on T. If F is locally uniformly convex, and if $\mathbf{E} \in (DGL^p | \mathbf{E}^{(p)}_{\rho,F})$ relative to T for some $1 \le p < \infty$, then $\mathbf{E}^{(p)}_{\rho,F}$ is locally uniformly convex.*

Proof. It follows immediately from the local uniform convexity of F that the norm on F is strictly monotone and by Corollary 2.4 and Proposition 2.6, it follows that the norm on F is order-continuous. Consequently, it follows from Theorem 5.1(i) (a), (c) that $\Phi_{\rho,F} \in (A)$ and $\Phi_{\rho,F} \in (SM|T)$. Theorem 5.1 (i)(e) implies that $\Phi_{\rho,F} \in (LUR_1|T)$. The assertion of the Theorem now follows from Theorem 4.2 and Theorem 4.3. □

Corollary 5.8. *Let $(F, \|\cdot\|_F)$ be a Banach function space on the measure space (T, μ) where T is either the interval $(0, \alpha)$ for some $0 < \alpha \le \infty$ and μ is Lebesgue measure m, or T is a countable subset $\{t_n\}_{n \in \mathbb{N}}$ and μ is counting measure. Assume that F is locally uniformly convex and let ρ be a strictly positive weight on T.*

(i) *If the Banach couple $\mathbf{E} = (E_0, E_1)$ is p-exact for some $1 < p < \infty$ and if each of E_0, E_1 is locally uniformly convex, then $\mathbf{E}^{(p)}_{\rho,F}$ is locally uniformly convex.*

(ii) *If $T = (0, \infty)$ and μ is Lebesgue measure m, if the Banach couple $\mathbf{E} = (E_0, E_1)$ is exact, and if each of E_0, E_1 is locally uniformly convex,* *then* $\mathbf{E}_{\rho,F}$ *is locally uniformly convex.*

Proof. By p-exactness of the couple $\mathbf{E}$ for some $1 < p < \infty$ and the local uniform convexity of each of the spaces E_0, E_1, it follows immediately from Theorem 3.13(i) that $\mathbf{E} \in (DGL^p|S(\mathbf{E}))$ relative to T. The assertion of (i) now follows immediately from Theorem 5.7. The assertion of (ii) follows similarly from Theorem 3.8(i) and Theorem 5.7. □

Remark. Let us note that it follows from Proposition 3.2 that the couple $\mathbf{E}$ is p-exact for every $1 \leq p < \infty$ if $\mathcal{T}$ is a Hausdorff linear topology on $S(\mathbf{E})$ which is weaker than the norm topology, and for which the Banach couple $\mathbf{E}$ is $\mathcal{T}$-closed. As an illustration of the utility of this remark, we now give an application to the embedding properties of a locally uniformly convex Banach space with an unconditional basis.

Corollary 5.9. *Every locally uniformly convex Banach space with an unconditional basis is isomorphic to a complemented subspace of a locally uniformly convex space with a* 1*-symmetric basis.*

Proof. We let $\mathbf{E}$ be the Banach couple (l_q, l_p) for some $1 < p < q < \infty$, and let $\{m_n\}_{n=1}^{\infty}$ be the sequence constructed in [23], Proposition 3.b.4. Let $a_n := m_n^{-1}$, $b_n := m_n$, $n = 1, 2, \dots$. We let Y be a locally uniformly convex space with a 1-unconditional normalised basis $\{e_n\}_{n=1}^{\infty}$. Let $T = \{t_n\}_{n=1}^{\infty}$ be the set $\{m_n^{-2}\}_{n=1}^{\infty}$, equipped with counting measure μ and let F be the Banach function space induced by Y on the measure space (T, μ). Note that F is locally uniformly convex. The weight ρ is defined on T by setting

$$\rho(t_n) = a_n = m_n^{-1}, \quad n = 1, 2, \dots .$$

We now define the Banach space X by setting

$$X := \mathbf{E}^{(2)}_{\rho,F} = \left\{ x \in l^q : \| \sum_{n=1}^{\infty} k_2(x, a_n, b_n) e_n \|_Y < \infty \right\}$$

where

$$k_2(x, a_n, b_n) = \inf \{ \left(a_n^2 \|x^0\|_{l_p}^2 + b_n^2 \|x^1\|_{l_q}^2 \right)^{1/2} : x = x^0 + x^1, x_0 \in l_p, x^1 \in l_q \}.$$

Since $1 < p,\ q < \infty$, each of the spaces l_p, l_q is locally uniformly convex, and each unit ball is sequentially compact for the topology $\mathcal{T}$ of pointwise convergence on l^1. Consequently, the couple $\mathbf{E}$ is $\mathcal{T}$ -closed and so from Corollary 5.8 and the remark immediately following, it follows that the Banach space X is locally uniformly convex. Since X is a 1-interpolation space, it follows as observed in [23], Lemma 3.b.3, that the unit vector basis in l^p is a 1-symmetric basis in X. Finally [23] Proposition 3.b.4 implies that X contains a complemented subspace isomorphic to the space Y. □

Let us remark that it does not appear that the result of the preceding Corollary follows from a direct application of the main result of [14] that every order continuous Banach lattice admits an equivalent locally uniformly convex norm, since application of this result to the space X does not guarantee that the basis remains 1-symmetric.

Theorem 5.10. *Let G be a fully symmetric Banach function space on $(0,\alpha)$, $0 < \alpha \le \infty$. Suppose that G is locally uniformly convex and that $\mathbf{E} \in (DGL|\mathbf{E}_G)$ relative to $T = (0,\alpha)$.*

(i) *If $\alpha < \infty$, or, if $\alpha = \infty$ and $G \nsubseteq L_1$, then $\mathbf{E}_G$ is locally uniformly convex.*
(ii) *If $\alpha = \infty$ and if $E_1 \subseteq E_0$ then $\mathbf{E}_G$ is locally uniformly convex.*
(iii) *If $\alpha = \infty$ and $\mathbf{E} \in (D^\infty)$, then $\mathbf{E}_G$ is locally uniformly convex.*

Proof. (i) The assumption that G is locally uniformly convex implies that G has the Kadec-Klee property for the weak topology by Corollary 2.4 and that the norm on G is order-continuous by Proposition 2.2. It follows from [8] Proposition 2.1 that the norm on G is strictly K-monotone and so from Theorem 5.1(ii)(b) it follows that $\Phi_G \in (SM|\mathbb{R}^+)$. If $0 < \alpha < \infty$, or from the assumption that $G \nsubseteq L_1$ in the case that $\alpha = \infty$, the fact that G has order continuous norm implies via from Theorem 4.2 that $\Phi_G \in (A)$. From Theorem 5.1(ii)(d), it follows that $\Phi_G \in (LUR_0|\mathbb{R}^+)$. That $\mathbf{E}_G$ is locally uniformly convex in cases now follows from Theorem 4.2.
(ii) The proof of this assertion is almost identical to the proof of Theorem 5.6(ii) with an appeal to the proof of Theorem 4.2 rather than to that of Theorem 4.1. We omit the details.
(iii) The proof of this assertion is almost identical to the proof of Theorem 5.6(iii) by again appealing to the proof of Theorem 4.2 instead of that of Theorem 4.1 and noting that the assumption that G is locally uniformly convex implies that G has the Kadec-Klee property for local convergence in measure by Corollary 2.5. We again omit the details. □

Corollary 5.11. *Let G be a fully symmetric Banach function space on $[0,\infty)$. Suppose that G is locally uniformly convex and that the Banach couple $\mathbf{E}$ is exact. If E_0, E_1 are locally uniformly convex and if (i) $G \nsubseteq L_1$ or (ii) $E_1 \subseteq E_0$ or (iii) $\mathbf{E} \in (D^\infty)$, then $\mathbf{E}_G$ is locally uniformly convex.*

Proof. The given assumptions on E_0, E_1 imply via Theorem 3.8(i) that $\mathbf{E} \in (DGL|S(\mathbf{E}))$ relative to $(0,\infty)$. This implies immediately that $\mathbf{E} \in (DGL|\mathbf{E}_G)$ relative to $(0,\infty)$. The Corollary now follows immediately from Theorem 5.10. □

6. Appendix A

In this section, we consider in more detail the question of order continuity of K-interpolation functionals on the cone Q. In particular, we shall prove Theorem 5.2 which establishes conditions under which order continuity on Q of K-functionals of type Φ_G for some fully symmetric Banach function space G can be deduced from order continuity of the norm on the underlying parameter space G.

Proposition 6.1. *If Φ is a K-interpolation functional on Q, then the following statements are equivalent.*

(i) $\Phi \in (A)$.

(ii) *For every $f \in Q$ with $\Phi(f) < \infty$,*

$$\text{(a)} \quad \lim_{s\to 0} \Phi\left(\min(f, s\mathbf{1})\right) = 0, \quad \text{(b)} \quad \lim_{s\to\infty} \Phi\left(\min(f, \cdot/s)\right) = 0.$$

Proof. The proof of the implication (i)⇒(ii) is trivial. Assume now that the assertion of (ii) holds. Let $f,\ f_n \in Q,\ n = 1, 2, \ldots$, with $\Phi(f) < \infty$ and suppose that $f_n(t) \to 0$ for every $t > 0$. Let $\epsilon > 0$ be given. By (ii), there exist $s_1, s_2 > 0$ such that

$$\Phi\left(\min(f, s_1\mathbf{1})\right) < \epsilon/2, \quad \Phi\left(\min(K(f, \cdot/s_2)\right) < \epsilon/2.$$

Since $f_n(1) \to 0$, there exists a natural number $N = N(s_1, s_2)$ such that

$$f_n(1) \leq \min(s_1, 1/s_2)$$

for all $n \geq N$. Since $f_n \in Q,\ n = 1, 2, \ldots$, it follows that

$$f_n(t) \leq \begin{cases} f_n(1), & \text{for } 0 < t \leq 1; \\ f_n(1)t, & \text{for } t \geq 1. \end{cases}$$

and so

$$f_n(t) \leq f_n(1) + f_n(1)t \leq s_1 + t/s_2$$

for all $t > 0$ and for all $n \geq N$. We obtain that

$$\begin{aligned} \Phi\left(\min(f, f_n)\right) &\leq \Phi\left(\min(f, s_1\mathbf{1} + \cdot/s_2)\right) \\ &\leq \Phi\left(\min(f, s_1\mathbf{1})\right) + \Phi\left(\min(f, \cdot/s_2)\right) \\ &< \epsilon/2 + \epsilon/2 = \epsilon, \end{aligned}$$

for all $n \geq N$. From this, it follows that $\Phi \in (A)$. □

Lemma 6.2. *Let Φ be a K-interpolation functional on Q.*

(i) *If there exists* $f^0 \in Q$ *and a positive constant* $C_0 > 0$ *such that* $\Phi(f^0) < \infty$ *and* $\lim_{t\to 0+} f^0(t) > C_0$, *then*

$$\lim_{s\to 0} \Phi\left(\min(f, s\mathbf{1})\right) = 0$$

for all $f \in Q$ *with* $\Phi(f) < \infty$..

(ii) *(ii) If there exists* $f^0 \in Q$ *and a positive constant* $C_1 > 0$ *such that* $\Phi(f^0) < \infty$ *and* $\lim_{t\to\infty} f^0(t)/t > C_1$, *then*

$$\lim_{s\to\infty} \Phi\left(\min(f, \cdot/s)\right) = 0$$

for all $f \in Q$ *with* $\Phi(f) < \infty$.

Proof. (i) Since f^0 is increasing, we have that

$$f^0(t) \geq \lim_{t\to 0} f^0(t) > C_0$$

for all $t > 0$. Consequently, if $f \in Q$ and $\Phi(f) < \infty$, then

$$0 \leq \lim_{s\to 0} \Phi\left(\min(f, s\mathbf{1})\right) \leq \lim_{s\to 0} \Phi\left(\min(f, sf^0/C_0)\right) \leq \lim_{s\to 0} s\Phi(f^0))/C_0 = 0.$$

(ii) Since $\lim_{t\to\infty} f^0(t)/t > C_1 > 0$, there exists $a > 0$ such that $f^0(t)/t > C_1$ for all $t \geq a$. For $0 < t \leq a$, we have that

$$f^0(t)/t \geq f^0(a)/a > C_1$$

and so $f^0(t)/t > C_1$ for all $t > 0$. Consequently, if $f \in Q$ with $\Phi(f) < \infty$, then

$$0 \leq \lim_{s\to\infty} \Phi\left(\min(f, \cdot/s)\right) \leq \lim_{s\to\infty} \Phi\left(\min(f, f^0/C_1 s)\right) \leq \lim_{s\to\infty} \Phi\left(f^0\right)/C_1 s = 0.$$

□

We will say that the K-interpolation functional Φ is *normal* on Q if and only if

$$f_n \in Q,\ n \in \mathbb{N}, \quad \Phi\left(\sum_{n=1}^{\infty} f_n\right) < \infty \Longrightarrow \Phi(f_n) \to 0.$$

Lemma 6.3. *Let* Φ *be a* K*-interpolation functional on* Q, *let* $f \in Q$ *and suppose that* $\Phi(f) < \infty$. *For each* $s > 0$, *let* $f_s \in Q$ *be defined by setting*

$$f_s(t) := \min(f(s), f(s)t/s), \quad t > 0,$$

and let Φ *be normal on* Q.

(i) *If* $\lim_{t\to 0} f(t) = 0$ *and* $\lim_{t\to 0} f(t)/t = \infty$, *then* $\lim_{s\to 0} \Phi(f_s) = 0$.

(ii) *If* $\lim_{t\to\infty} f(t) = \infty$ *and* $\lim_{t\to\infty} f(t)/t = 0$, *then* $\lim_{s\to\infty} \Phi(f_s) = 0$.

Proof. Suppose that there exists a sequence $s_n \downarrow 0$ and $\epsilon > 0$ such that

$$\Phi(f_{s_n}) = \Phi\left(\min(f(s_n)\mathbf{1}, f(s_n)\cdot/s_n)\right) > \epsilon, \quad n = 1, 2, \dots \quad .$$

Using the stated assumptions, we may assume without loss of generality that

$$f(s_{n+1}) < f(s_n)/2 \quad \text{and} \quad f(s_n)/s_n < (1/2)f(s_{n+1})/s_{n+1}, \quad n = 1, 2, \dots \ .$$

We set

$$g(t) := \sum_{n=1}^{\infty} \min(f(s_n), tf(s_n)/s_n), \quad t > 0.$$

If $t \in (s_{n+1}, s_n)$ then

$$g(t) = \left(\sum_{k=1}^{n} f(s_k)/s_k\right) t + \sum_{k=n+1}^{\infty} f(s_k).$$

In particular the function g is linear on each interval $[s_{n+1}, s_n]$. Further,

$$\begin{aligned} g(s_n) &= \left(\sum_{k=1}^{n-1} f(s_k)/s_k\right) s_n + f(s_n) + \sum_{k=n+1}^{\infty} f(s_k) \\ &\leq \sum_{k=1}^{n-1} f(s_n)/2^{n-k} + f(s_n) + \sum_{k=n+1}^{\infty} f(s_n)/2^{k-n} \\ &< 3f(s_n), \quad n = 1, 2, \dots \ . \end{aligned}$$

Since f is concave and g is linear on each interval $[s_{n+1}, s_n]$, it now follows that $g(t) < 3f(t)$ for all $t > 0$ and so

$$\Phi(g) = \Phi\left(\sum_{n=1}^{\infty} \min(f(s_n)\mathbf{1}, f(s_n)\cdot/s_n)\right) < 3\Phi(f) < \infty \ .$$

Since Φ is normal, it follows that

$$\Phi\left(\min(f(s_n)\mathbf{1}, f(s_n)\cdot/s_n)\right) \to 0$$

and this is a contradiction. This proves (i). The proof of (ii) is similar and the details are omitted. □

Proof of Theorem 5.2. Let G be a fully symmetric Banach function space on $[0, \alpha)$, $0 < \alpha \leq \infty$, with order continuous norm. Let $f \in Q$ satisfy $\Phi_G(f) < \infty$. From the definition of Φ_G, it follows that $f \in Q_0$ so that

$\lim_{t\to 0} f(t) = 0$. We show first that

$$\lim_{s\to 0} \Phi_G(\min(f, s\mathbf{1})) = 0.$$

If we observe that

$$\Phi_G(\min(f, s\mathbf{1})) = \|f'\chi_{[0,\tau)}\|_G$$

where $\tau = \inf\{t : f(t) = s\}$, then it follows from the fact that $\lim_{t\to 0} f(t) = 0$ and from the assumption that the norm on G is order continuous, that

$$\lim_{s\to 0} \Phi_G(\min(f, s\mathbf{1})) = \lim_{\tau\to 0} \|f'\chi_{[0,\tau)}\|_G = 0.$$

We now show that

$$\lim_{s\to\infty} \Phi_G(\min(f, \cdot/s)) = 0.$$

In the case that $\alpha < \infty$, this is an immediate consequence of the fact that

$$[\min(f, \cdot/s)]' = \chi_{[0,s)}/s$$

for all sufficiently large s. We may suppose then that $\alpha = \infty$. By Lemma 6.2, we may assume that $\lim_{t\to\infty} f(t)/t = 0$. Assume first that $\lim_{t\to\infty} f(t) = \infty$. We remark that the assumption that the norm on G is order continuous together with an application of the well-known differentiation theorem of Fubini (see, for example [26],Chapter 1.5) implies that the functional Φ_G is normal. It then follows from Lemma 6.3 (ii) that

$$\begin{aligned}
\lim_{s\to\infty} \|(f(s)/s)\chi_{[0,s)}\|_G &= \lim_{s\to\infty} \|\frac{d}{dt}\min(f(s)\mathbf{1}, f(s)\cdot/s))\|_G \\
&= \lim_{s\to\infty} \Phi_G(\min(f(s)\mathbf{1}, f(s)\cdot/s)) = 0.
\end{aligned}$$

If now $s > 0$ is given, let $\sigma = \max\{\tau : 1/s = f(\tau)/\tau\}$ and observe that

$$\Phi_G(\min(f, \cdot/s)) = \|(f(\sigma)/\sigma)\chi_{[0,\sigma)} + f'\chi_{[\sigma,\infty)}\|_G.$$

From this it follows that

$$\lim_{s\to\infty} \Phi_G(\min(f, \cdot/s)) \le \lim_{\sigma\to\infty} \|(f(\sigma)/\sigma)\chi_{[0,\sigma)}\|_G + \lim_{\sigma\to\infty} \|f'\chi_{[\sigma,\infty)}\|_G = 0,$$

again using order continuity of the norm on G.
Finally, assume that $\lim_{t\to\infty} f(t) < \infty$. From the assumption that $G \not\subseteq L_1[0,\infty)$, there exists $g \in Q_0$ with $g' \in E$ such that $\lim_{t\to\infty} g(t) = \infty$. from the first part of the proof, it follows that

$$\lim_{s\to\infty} \Phi_G(\min(g, \cdot/s) = 0.$$

However,

$$\Phi_G(\min(f, \cdot/s)) \leq \Phi_G(\min(g, \cdot/s)),$$

for all sufficiently large s. This implies that

$$\lim_{s\to\infty} \Phi_G(\min(f, \cdot/s)) = 0$$

and this suffices to complete the proof of the Theorem via Proposition 6.1. □

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